

AN EQUIVALENCE PRINCIPLE BETWEEN POLYNOMIAL AND SIMULTANEOUS DIOPHANTINE APPROXIMATION

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ABSTRACT. We show that Mahler's classification of real numbers ζ with respect to the growth of the sequence $(w_n(\zeta))_{n \geq 1}$ is equivalently induced by certain natural assumptions on the decay of the sequence $(\lambda_n(\zeta))_{n \geq 1}$ concerning simultaneous rational approximation. Thereby we obtain a much clearer picture on simultaneous approximation to successive powers of a real number in general. Another variant of the Mahler classification concerning uniform approximation by algebraic numbers is shown as well. Our method has several applications to classic exponents of Diophantine approximation and metric theory. We deduce estimates on the Hausdorff dimension of well-approximable vectors on the Veronese curve and refine the best known upper bound for the exponent $\hat{\lambda}_n(\zeta)$ for even $n \geq 4$.

Keywords: exponents of Diophantine approximation, Mahler's classification, parametric geometry of numbers

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1. CLASSICAL EXPONENTS OF DIOPHANTINE APPROXIMATION

In this paper we establish a link between classical intensely studied Diophantine approximation problems. Let ζ be a transcendental real number and m, n be positive integers. On one hand, we are concerned with approximation small polynomial evaluations $|P(\zeta)|$ for integer polynomials P of degree at most n , in terms of the height of P . This problem is known to be closely connected to approximation to ζ by real algebraic numbers of degree at most n . On the other hand, we deal with simultaneous rational approximation to $(\zeta, \zeta^2, \dots, \zeta^m)$. We establish connections between these classical problems, for suitable pairs m, n . This will lead to a better understanding of both classical problems on their own. Mahler introduced the classical exponent $w_n(\zeta)$ as the supremum of real numbers w such that

$$(1) \quad 0 < |P(\zeta)| \leq H(P)^{-w},$$

has infinitely many solutions $P \in \mathbb{Z}[T]$ of degree at most n . Here $H(P)$ is the maximum modulus of the coefficients of P . Bugeaud and Laurent defined the uniform exponent $\hat{w}_n(\zeta)$ as the supremum of $w \in \mathbb{R}$ such that the system

$$(2) \quad H(P) \leq X, \quad 0 < |P(\zeta)| \leq X^{-w},$$

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has a solution $P \in \mathbb{Z}[T]$ of degree at most n for all large X . It is easy to see the definition of $w_n(\zeta)$ is equivalent to requiring (2) to be satisfied for certain arbitrarily large X . By Dirichlet's Theorem we have

$$(3) \quad w_n(\zeta) \geq \widehat{w}_n(\zeta) \geq n, \quad n \geq 1.$$

Moreover, since we admit more polynomials as n increases we obviously have

$$(4) \quad w_1(\zeta) \leq w_2(\zeta) \leq \cdots, \quad \widehat{w}_1(\zeta) \leq \widehat{w}_2(\zeta) \leq \cdots.$$

In this paper we focus on the best approximation exponents w_n , however we contribute some results involving the uniform exponent \widehat{w}_n in Section 4.3. It is conjectured that (3) and (4) are the only limitations on sequences $(w_n(\zeta))_{n \geq 1}$ if we may choose any transcendental real ζ . We recall this partial assertion of the *Main problem* in [10, Section 3.4, page 61] on the *joint spectrum of $(w_n(\zeta))_{n \geq 1}$* .

Problem 1. Let $(w_n)_{n \geq 1}$ be a non-decreasing sequence of real numbers with $w_n \geq n$. Does there exist ζ such that $w_n(\zeta) = w_n$ simultaneously for all $n \geq 1$?

A positive answer is strongly expected, but only special cases have been verified. Mahler classified the transcendental real numbers in terms of the growth of the sequence $(w_n(\zeta))_{n \geq 1}$. He called a transcendental real number ζ a U_m -number if $w_m(\zeta) = \infty$ and m is the smallest such index. The set of U -numbers is defined as the disjoint union of the sets of U_m -numbers over $m \geq 1$. Due to Mahler, a number ζ is called T -number if $w_n(\zeta) < \infty$ for all $n \geq 1$, but $\limsup_{n \rightarrow \infty} w_n(\zeta)/n = \infty$ holds. Finally, the remaining numbers for which $w_n(\zeta)/n \ll 1$ are called S -numbers. A famous result of Sprindžuk [37] claims that almost all numbers in sense of Lebesgue measure are S -numbers. Bernik [9] refined this in a metrical sense by showing that

$$(5) \quad \dim(\{\zeta \in \mathbb{R} : w_n(\zeta) \geq w\}) = \dim(\{\zeta \in \mathbb{R} : w_n(\zeta) = w\}) = \frac{n+1}{w+1}, \quad w \geq n.$$

Here and in the sequel \dim denotes the Hausdorff dimension, see [18] for an introduction. We remark that Baker and Schmidt [5] had known the lower bound before. By (5) the sets of U -numbers and T -numbers in fact both have dimension 0. However, they are well-known to be non-empty, see LeVeque [23] and Schmidt [33]. We refer to [10] for more results connected to Mahler's classification and related topics.

We want to relate the exponents $w_n(\zeta)$ to the exponents of simultaneous approximation introduced by Bugeaud and Laurent [13]. They define the exponent $\lambda_n(\zeta)$ as the supremum of $\lambda \in \mathbb{R}$ such that the system

$$(6) \quad 1 \leq |x| \leq X, \quad \max_{1 \leq i \leq n} |\zeta^i x - y_i| \leq X^{-\lambda},$$

has a solution $(x, y_1, y_2, \dots, y_n) \in \mathbb{Z}^{n+1}$ for arbitrarily large values of X . Similarly, they denote by $\widehat{\lambda}_n(\zeta)$ the supremum of λ such that (6) has a solution for all $X \geq X_0$. For any transcendental real ζ and $n \geq 1$, by Dirichlet's Theorem these exponents satisfy

$$(7) \quad \lambda_n(\zeta) \geq \widehat{\lambda}_n(\zeta) \geq \frac{1}{n}.$$

Moreover from the definition we see that

$$(8) \quad \lambda_1(\zeta) \geq \lambda_2(\zeta) \geq \cdots, \quad \widehat{\lambda}_1(\zeta) \geq \widehat{\lambda}_2(\zeta) \geq \cdots.$$

Khintchine [21] was the first to show a relation between polynomial and simultaneous approximation. His transference principle asserts

$$(9) \quad \frac{w_n(\zeta)}{(n-1)w_n(\zeta) + n} \leq \lambda_n(\zeta) \leq \frac{w_n(\zeta) - n + 1}{n}.$$

So far, essentially the transference principle has been the only tool for comparing the sequences $(w_n(\zeta))_{n \geq 1}$ and $(\lambda_n(\zeta))_{n \geq 1}$. It provides very limited information. In particular it is far from clear what to expect for the joint behavior of the sequences $(\lambda_n(\zeta))_{n \geq 1}$, in contrast to the precise conjecture in Problem 1. In Section 4.1 below we will quote the few previously known results. Our main result Theorem 2.1 will provide very much refined information on the interaction of the two sequences $(w_n(\zeta))_{n \geq 1}$ and $(\lambda_n(\zeta))_{n \geq 1}$ for given ζ , and thereby lead to a much better understanding of the latter sequence. Roughly speaking our method is based on comparison of $w_n(\zeta)$ with $\lambda_m(\zeta)$ for suitable pairs m, n . In fact we usually take m to be reasonably larger than n , in contrast to $m = n$ in (9). We emphasize that for our method it is crucial that we deal with successive powers of a real number, whereas (9) can be adapted for vectors $\underline{\zeta} \in \mathbb{R}^n$ that are \mathbb{Q} -linearly independent together with $\{1\}$.

2. EQUIVALENCE PRINCIPLES

2.1. Simultaneous approximation. In our main result we link the sequences of exponents $(w_n(\zeta))_{n \geq 1}$ and $(\lambda_n(\zeta))_{n \geq 1}$.

Theorem 2.1 (Equivalence principle I). *Let ζ be a transcendental real number. Then ζ is a U -number if and only if*

$$(10) \quad \lim_{n \rightarrow \infty} \lambda_n(\zeta) > 0.$$

More precisely, if ζ is a U_m -number, then $\lambda_n(\zeta) = \frac{1}{m-1}$ for all sufficiently large n . Moreover, ζ is a T -number if and only if

$$\lim_{n \rightarrow \infty} \lambda_n(\zeta) = 0, \quad \limsup_{n \rightarrow \infty} n\lambda_n(\zeta) = \infty.$$

Finally ζ is an S -number if and only if

$$\limsup_{n \rightarrow \infty} n\lambda_n(\zeta) < \infty.$$

The theorem shows that Mahler's classification can be equivalently obtained by natural assumptions on the decay of the sequence $(\lambda_n(\zeta))_{n \geq 1}$. The transference principle (9) admits the conclusion $\lim_{n \rightarrow \infty} \lambda_n(\zeta) = 0$ upon $\liminf_{n \rightarrow \infty} w_n(\zeta)/n = 1$, a reasonably stronger condition than ζ being no U -number. Hence (9) does not rule out (10) even for certain S -numbers. As a first corollary we determine all limits of the sequences $(\lambda_n(\zeta))_{n \geq 1}$.

Corollary 2.2. *The set of all values $\lim_{n \rightarrow \infty} \lambda_n(\zeta)$ as ζ attains any transcendental real number is precisely the countable set $\{0, \infty\} \cup_{n \geq 1} \{1/n\}$.*

Proof. For any S -number and T -number the limit is 0 by Theorem 2.1. For a U_m -number the limit is $1/(m-1)$ again by Theorem 2.1. The claim follows. \square

Remark 1. Previous results recalled in Section 4.1 below could have settled $\{0, 1, \infty\} \subseteq \mathcal{S} \subseteq [0, 1] \cup \{\infty\}$ for \mathcal{S} the set of limits as in the corollary. Indeed, the inclusion $\{0, 1, \infty\} \subseteq \mathcal{S}$ follows from Sprindžuk [37] and Bugeaud [11, Theorem 4, Corollary 2], whereas the consequence [28, Corollary 1.9] of (38) implies $(1, \infty) \cap \mathcal{S} = \emptyset$.

We can provide effective relations between the sequences $(w_n(\zeta))_{n \geq 1}$ and $(\lambda_n(\zeta))_{n \geq 1}$. We recall the notion of the order $\tau(\zeta)$ of T -number [10], defined as

$$\tau(\zeta) = \limsup_{n \rightarrow \infty} \frac{\log w_n(\zeta)}{\log n}.$$

We have $\tau(\zeta) \in [1, \infty]$ for any T -number ζ by (3). All so far constructed T -numbers have order $\tau(\zeta) \geq 3$, and R. Baker [6] conversely constructed T -numbers of the given degree $\tau(\zeta) \in [3, \infty]$. See also [10, Theorem 7.2], however there seems to be a problem in the proof as in (7.28) a stronger version than the assumed (7.24) is used. A positive answer to Problem 1 would imply that T -numbers of any degree $\tau(\zeta) \in [1, \infty]$ exist. We propose a somehow dual order $\sigma(\zeta)$, which we define as

$$\sigma(\zeta) = \limsup_{n \rightarrow \infty} \frac{\log \lambda_n(\zeta)}{\log n}.$$

It follows from (7) and (8) that $\sigma(\zeta) \in [-1, 0]$ for any ζ which is not a Liouville number (i.e a U_1 -number). In fact even $\log \lambda_n(\zeta)/\log n \leq 0$ for all large n by [28, Theorem 1.6]. Further define

$$(11) \quad \overline{w}(\zeta) := \limsup_{n \rightarrow \infty} \frac{w_n(\zeta)}{n}, \quad \overline{\lambda}(\zeta) := \limsup_{n \rightarrow \infty} n\lambda_n(\zeta),$$

and

$$(12) \quad \underline{w}(\zeta) := \liminf_{n \rightarrow \infty} \frac{w_n(\zeta)}{n}, \quad \underline{\lambda}(\zeta) := \liminf_{n \rightarrow \infty} n\lambda_n(\zeta).$$

The set of S -numbers is defined as the set of numbers with $\overline{w}(\zeta) < \infty$. For S -numbers and T -numbers of degree $\tau(\zeta) = 1$, the quantities $\overline{w}(\zeta), \underline{w}(\zeta)$ provide a refined measure. Similarly $\overline{\lambda}(\zeta), \underline{\lambda}(\zeta)$ refine $\sigma(\zeta)$. We obtain connections between the quantities as follows.

Theorem 2.3. *For any real transcendental ζ we have*

$$(13) \quad \frac{(\overline{w}(\zeta) + 1)^2}{4\overline{w}(\zeta)} \leq \overline{\lambda}(\zeta) \leq \overline{w}(\zeta) + 2, \quad \frac{(\underline{w}(\zeta) + 1)^2}{4\underline{w}(\zeta)} \leq \underline{\lambda}(\zeta) \leq \underline{w}(\zeta) + 2,$$

and moreover

$$(14) \quad \sigma(\zeta) = -\frac{1}{\tau(\zeta)}.$$

In the theorem and generally in the sequel we always agree on $1/\infty = 0$ and $1/0 = +\infty$. There is no reason to believe that the bounds in (13) are optimal. It is tempting to conjecture that $\overline{w}(\zeta) = \overline{\lambda}(\zeta)$ and $\underline{w}(\zeta) = \underline{\lambda}(\zeta)$ hold for any transcendental real ζ .

2.2. Uniform approximation by algebraic numbers. In this section we establish another equivalence principle. We connect the Mahler classification with exponents of uniform approximation to a real number by algebraic numbers of degree bounded by some n . Let $w_n^*(\zeta)$ and $\widehat{w}_n^*(\zeta)$ be the supremum of w^* such that

$$(15) \quad |\zeta - \alpha| \leq H(\alpha)^{-1} X^{-w^*}$$

has a real algebraic solution α of degree at most n , for arbitrarily large and all large X , respectively. Here $H(\alpha) = H(P)$ for P the (up to sign) unique minimal polynomial of α with coprime integral coefficients. These exponents are closely linked to the polynomial exponents $w_n(\zeta), \widehat{w}_n(\zeta)$. In particular, the same partition of the transcendental real numbers is induced by replacing w_n in the Mahler classification above by w_n^* , as proposed by Koksma. Indeed this is an immediate consequence of the left estimate in

$$(16) \quad w_n^*(\zeta) \leq w_n(\zeta) \leq w_n^*(\zeta) + n - 1.$$

from [10, Lemma A8]. The analogous estimates hold for the uniform exponents, and together with upper bounds by Davenport and Schmidt [17] we may comprise

$$(17) \quad \widehat{w}_n^*(\zeta) \leq \widehat{w}_n(\zeta) \leq \min\{2n - 1, \widehat{w}_n^*(\zeta) + n - 1\}.$$

The bound $2n - 1$ was in fact slightly improved recently by Bugeaud and the author [15]. We show that the Mahler classification is obtained as well by natural assumptions on the sequence of uniform exponents $\widehat{w}_n^*(\zeta)$.

Theorem 2.4 (Equivalence principle II). *Let ζ be a transcendental real number. Then ζ is a U -number if and only if*

$$(18) \quad \lim_{n \rightarrow \infty} \widehat{w}_n^*(\zeta) < \infty.$$

More precisely, if ζ is a U_m -number, then $\widehat{w}_n^(\zeta) \in [m - 1, m]$ for all sufficiently large n . Moreover, ζ is a T -number if and only if*

$$(19) \quad \lim_{n \rightarrow \infty} \widehat{w}_n^*(\zeta) = \infty, \quad \liminf_{n \rightarrow \infty} \frac{\widehat{w}_n^*(\zeta)}{n} = 0.$$

Finally ζ is an S -number if and only if there exists a constant $\delta > 0$ such that $\widehat{w}_n^(\zeta) \geq \delta n$ for all $n \geq 1$.*

Remark 2. Several variants of equivalence principle II can be derived similarly. For example one can fix the degree of the algebraic numbers in (15) equal to n , or restrict to approximation by algebraic integers or algebraic units. See for example [16], [17], or [32].

Define the quantities

$$\underline{\widehat{w}}^*(\zeta) = \liminf_{n \rightarrow \infty} \frac{\widehat{w}_n^*(\zeta)}{n}, \quad \overline{\widehat{w}}^*(\zeta) = \limsup_{n \rightarrow \infty} \frac{\widehat{w}_n^*(\zeta)}{n},$$

and further let

$$\theta(\zeta) = \liminf_{n \rightarrow \infty} \frac{\log \widehat{w}_n^*(\zeta)}{\log n}.$$

By (17) we have $0 \leq \underline{\widehat{w}}^*(\zeta) \leq \overline{\widehat{w}}^*(\zeta) \leq 2$ and $\theta(\zeta) \in [0, 1]$. An effective version of the second equivalence principle reads as follows.

Theorem 2.5. *Let ζ be any transcendental real number. We have*

$$(20) \quad \frac{1}{\overline{w}(\zeta) + 2} \leq \widehat{w}^*(\zeta) \leq \min \left\{ \underline{w}(\zeta), \frac{4}{\overline{w}(\zeta)} \right\},$$

and

$$(21) \quad \frac{1}{\underline{w}(\zeta) + 2} \leq \overline{w}^*(\zeta) \leq \min \left\{ \overline{w}(\zeta), \frac{4}{\underline{w}(\zeta)} \right\}.$$

Moreover

$$(22) \quad \theta(\zeta) = \frac{1}{\tau(\zeta)} = -\sigma(\zeta).$$

It is somehow remarkable that for large values of $\overline{w}(\zeta)$ and $\underline{w}(\zeta)$, the respective lower and upper bound differ roughly by the same factor 4 as in Theorem 2.3. This is surprising as the proof of upper bounds in Theorem 2.5 is unrelated to Theorem 2.3. It is hard to predict if this factor 4 has any deeper meaning. Note that for the similarly defined quantities $\underline{w}^*(\zeta)$, $\overline{w}^*(\zeta)$, Wirsing's [38] estimate $w_n^*(\zeta) \geq (w_n(\zeta) + 1)/2 \geq (n + 1)/2$ and (16) imply

$$\frac{1}{2} \leq \max \left\{ \frac{\overline{w}(\zeta)}{2}, \overline{w}(\zeta) - 1 \right\} \leq \overline{w}^*(\zeta) \leq \overline{w}(\zeta),$$

and

$$\frac{1}{2} \leq \max \left\{ \frac{\underline{w}(\zeta)}{2}, \underline{w}(\zeta) - 1 \right\} \leq \underline{w}^*(\zeta) \leq \underline{w}(\zeta).$$

Thus $\tau(\zeta)$ equals the order $\tau^*(\zeta)$ obtained by replacing $w_n(\zeta)$ by $w_n^*(\zeta)$. Hence a variant of Theorem 2.5, in terms of quantities derived from $w_n^*(\zeta)$ and $\widehat{w}_n^*(\zeta)$ only, can be formulated. We do not explicitly state it.

Similar to Corollary 2.2, we can ask for the set \mathscr{W} of limits of the sequences $(\widehat{w}_n^*(\zeta))_{n \geq 1}$ as ζ attains every real number. We conjecture that $\mathscr{W} = \{\infty\} \cup \{1, 2, 3, \dots\}$. However, Theorem 2.4 only admits the inclusion $\mathscr{W} \supseteq \{1, \infty\}$, and conversely we cannot even exclude $\mathscr{W} = [1, \infty]$.

We recapitulate that in Section 2 we derived four equivalent definitions of the Mahler classification in terms of the sequences $(w_n(\zeta))_{n \geq 1}$, $(\lambda_n(\zeta))_{n \geq 1}$, $(w_n^*(\zeta))_{n \geq 1}$ and $(\widehat{w}_n^*(\zeta))_{n \geq 1}$ respectively. It is natural to ask if the sequences $(\widehat{w}_n(\zeta))_{n \geq 1}$ and $(\widehat{\lambda}_n(\zeta))_{n \geq 1}$ can be somehow included in the picture. However, almost all S -numbers satisfy $\widehat{w}_n(\zeta) = n$ and $\widehat{\lambda}_n(\zeta) = 1/n$ for all $n \geq 1$ by Sprindžuk [37], and any Liouville number (i.e. a U_1 -number) shares the same property by [28, Corollary 5.2]. Hence it seems not to be possible.

We give a brief outline of the upcoming sections. In Section 3 below we establish several partial results, which combine to Theorem 2.1 and Theorem 2.3. These partial results have additional interesting consequences on their own, gathered in Section 4. There we refine the upper bound for the uniform exponents $\widehat{\lambda}_n(\zeta)$ for even n . Furthermore we study the consequences of the equivalence principle to the metric problem of determining the Hausdorff dimension of vectors on the Veronese curve that are simultaneously approximable to a given degree. Moreover, for numbers ζ that admit many very small evaluations at integer polynomials of bounded degree, we provide a rate of decay for the

exponents $\lambda_n(\zeta)$ for large n . Suitable numbers include the Champerowne number and any number with $\widehat{w}_n(\zeta) > n$ for some $n \geq 2$. The proofs, unless reasonably short, are carried out in Section 5.

3. REFINEMENTS OF THE EQUIVALENCE PRINCIPLE

Theorem 2.1 will be an immediate consequence of Theorem 3.1, Theorem 3.3 and Theorem 3.4 formulated in this Section 3.

3.1. Upper bounds for λ_n . The upper bounds in Theorem 2.1 and Theorem 2.3 are a consequence of the following very general Theorem 3.1. We agree on $w_0(\zeta) = 0$.

Theorem 3.1. *Let $n \geq 1$ be an integer and ζ a transcendental real number. Assume $w_n(\zeta) < \infty$. Then we have*

$$(23) \quad \lambda_N(\zeta) \leq \max \left\{ \frac{1}{\widehat{w}_n(\zeta)}, \frac{1}{\widehat{w}_{N-n+1}(\zeta) - w_n(\zeta)} \right\}, \quad N \geq \lceil w_n(\zeta) \rceil + n - 1.$$

Moreover, in case of $w_n(\zeta) < 2n + 1$ we have

$$(24) \quad \lambda_N(\zeta) \leq \max \left\{ \frac{1}{\widehat{w}_n(\zeta)}, \frac{1}{\widehat{w}_{N-n+1}(\zeta) - w_{N-2n}(\zeta)} \right\}, \quad \lfloor w_n(\zeta) \rfloor + n \leq N \leq 3n.$$

We see that in case of $w_n(\zeta) < 2n$, for $N < 3n$ the bound (24) is possibly stronger than (23) because of the smaller index in the right expression. The case $w_n(\zeta) < n + 1$ and $N = 2n$ in (24) will be crucial for the improved upper bounds for the exponents $\widehat{\lambda}_{2n}(\zeta)$ in Section 4.2. The estimate (23) with a suitable choice of N yields the desired implications for the equivalence principle.

Corollary 3.2. *Let $n \geq 1$ be an integer and ζ a transcendental real number and assume $w_n(\zeta) < \infty$. Then*

$$(25) \quad \lambda_N(\zeta) \leq \frac{1}{n}, \quad N \geq \lceil w_n(\zeta) \rceil + 2n - 1.$$

Thus, if ζ is not a U -number then $\lim_{n \rightarrow \infty} \lambda_n(\zeta) = 0$, and if ζ is an S -number then $\overline{\lambda}(\zeta) = \limsup_{n \rightarrow \infty} n\lambda_n(\zeta) < \infty$, and more precisely

$$(26) \quad \overline{\lambda}(\zeta) \leq \overline{w}(\zeta) + 2, \quad \underline{\lambda}(\zeta) \leq \underline{w}(\zeta) + 2.$$

Proof. In view of (3), as soon as $N \geq w_n(\zeta) + 2n - 1$ the right hand side in (23) can be estimated above by

$$\max \left\{ \frac{1}{\widehat{w}_n(\zeta)}, \frac{1}{\widehat{w}_{N-n+1}(\zeta) - w_n(\zeta)} \right\} \leq \max \left\{ \frac{1}{n}, \frac{1}{N - n + 1 - w_n(\zeta)} \right\} = \frac{1}{n}.$$

Hence (25) follows. The claim (26) follows by reversing the argument. For $\epsilon > 0$ and large N , choose $n = \lceil N/(\overline{w}(\zeta) + 2 + \epsilon) \rceil$ and $n = \lceil N/(\underline{w}(\zeta) + 2 + \epsilon) \rceil$, respectively. The condition in (25) is satisfied and we obtain $\lambda_N(\zeta) \leq 1/n = (\overline{w}(\zeta) + 2)/N + \epsilon_N$ and $\lambda_N(\zeta) \leq 1/n = (\underline{w}(\zeta) + 2)/N + \epsilon_N$, respectively, where ϵ_N tends to 0 as ϵ does and $N \rightarrow \infty$. It suffices to let $\epsilon \rightarrow 0$. \square

If $w_n(\zeta)$ is not too large (considerably smaller than $2n$) and for small t the values $w_t(\zeta)$ do not exceed t by much, then (24) yields smaller N for the conclusion in (25). On the other hand, the bound $1/n$ in (25) in general cannot be improved for any N , as follows from Theorem 2.1 by taking ζ a U_{n+1} -number. See also Corollary 57 in Section 4.3 below.

3.2. Lower bounds for λ_n . To formulate the results of this section in full extent, we need to define successive minima exponents that refine the classical exponents $w_n(\zeta)$ and $\lambda_n(\zeta)$. For $1 \leq j \leq n+1$, let $\lambda_{n,j}(\zeta)$ and $\widehat{\lambda}_{n,j}(\zeta)$ be the supremum of λ for which (6) has j linearly independent integer vector solutions for arbitrarily large X and all large X , respectively. Similarly, let $w_{n,j}(\zeta)$ and $\widehat{w}_{n,j}(\zeta)$ be the supremum of w for which (2) has j linearly independent polynomial solutions for arbitrarily large and all large X , respectively. Obviously, for $j = 1$ we recover the corresponding classical exponents, and the relations

$$\begin{aligned} \lambda_{n,1}(\zeta) &\geq \lambda_{n,2}(\zeta) \geq \cdots \geq \lambda_{n,n+1}(\zeta), & \widehat{\lambda}_{n,1}(\zeta) &\geq \widehat{\lambda}_{n,2}(\zeta) \geq \cdots \geq \widehat{\lambda}_{n,n+1}(\zeta), \\ w_{n,1}(\zeta) &\geq w_{n,2}(\zeta) \geq \cdots \geq w_{n,n+1}(\zeta), & \widehat{w}_{n,1}(\zeta) &\geq \widehat{w}_{n,2}(\zeta) \geq \cdots \geq \widehat{w}_{n,n+1}(\zeta), \end{aligned}$$

hold.

Theorem 3.3. *Let $m \geq 2$ be an integer and ζ be a U_m -number. Then*

$$(27) \quad \lambda_n(\zeta) \geq \frac{1}{m-1}, \quad n \geq 1.$$

If and only if additionally $w_{m-1}(\zeta) = m-1$ holds, then

$$(28) \quad \lambda_n(\zeta) = \lambda_{n,2}(\zeta) = \cdots = \lambda_{n,m}(\zeta) = \frac{1}{m-1}, \quad n \geq m-1.$$

If and only if moreover $w_1(\zeta) = w \in [1, 2]$ and $w_t = t$ for any $2 \leq t \leq m-1$, then the sequence $(\lambda_n(\zeta))_{n \geq 1}$ is given by

$$(29) \quad \left(w, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{m-2}, \frac{1}{m-1}, \frac{1}{m-1}, \frac{1}{m-1}, \dots \right).$$

Remark 3. Any U_m -number ζ satisfies $\widehat{w}_n^*(\zeta) \leq m$ for all $n \geq 1$, see [15, Corollary 2.5]. Combination with the well-known bound (71) below would yield $\lambda_n(\zeta) \geq 1/m$ for any U_m -number ζ and $n \geq 1$, a weaker conclusion than (27).

Remark 4. Obviously $n = m-1$ is the smallest index for which (28) can possibly hold by (8). Notice that Theorem 2.1 (or Theorem 3.1) yields (28) (for the first minimum exponent) for any U_m -number when n is sufficiently large.

Remark 5. We notice that (29) with $w = 1$ coincides with the sequence $(\lambda_n(\zeta))_{n \geq 1}$ of any real algebraic number ζ of degree exactly m , a well-known consequence of Schmidt's Subspace Theorem. Hence we have a criterion when a real number behaves like an algebraic real number of given degree with respect to simultaneous approximation.

If we agree on $1/0 = \infty$ then (27) is true for $n = 1$ as well, but has been observed in [11] as a consequence of (37). For $k \leq n$, the estimate (27) follows from Khintchine's inequality (9) and (8), however for $k > n$ the result is new. A similar method as in the proof of Theorem 3.3 will lead to the next partial claim of Theorem 2.1.

Theorem 3.4. *For any transcendental real ζ the quantities defined in (11), (12) satisfy*

$$(30) \quad \frac{(\overline{w}(\zeta) + 1)^2}{4\overline{w}(\zeta)} \leq \overline{\lambda}(\zeta), \quad \frac{(\underline{w}(\zeta) + 1)^2}{4\underline{w}(\zeta)} \leq \underline{\lambda}(\zeta).$$

In particular, any T -number ζ satisfies

$$(31) \quad \limsup_{n \rightarrow \infty} n\lambda_n(\zeta) = \infty.$$

We close this section with a variant of Theorem 3.4 for uniform exponents, and for sake of completeness also add consequences of Theorem 2.5 and from [29]. Define the quantities

$$(32) \quad \overline{\widehat{w}}(\zeta) := \limsup_{n \rightarrow \infty} \frac{\widehat{w}_n(\zeta)}{n}, \quad \overline{\widehat{\lambda}}(\zeta) := \limsup_{n \rightarrow \infty} n\widehat{\lambda}_n(\zeta),$$

and

$$(33) \quad \underline{\widehat{w}}(\zeta) := \liminf_{n \rightarrow \infty} \frac{\widehat{w}_n(\zeta)}{n}, \quad \underline{\widehat{\lambda}}(\zeta) := \liminf_{n \rightarrow \infty} n\widehat{\lambda}_n(\zeta).$$

They satisfy $1 \leq \underline{\widehat{w}}(\zeta) \leq \overline{\widehat{w}}(\zeta) \leq 2$ and $1 \leq \underline{\widehat{\lambda}}(\zeta) \leq \overline{\widehat{\lambda}}(\zeta) \leq 2$ by (3), (7), (17) and the estimate $\widehat{\lambda}_n(\zeta) \leq 2/n$, as established in [17, Theorem 2a], [22], [29] reproduced in Section 4.3 below. Moreover we have

$$(34) \quad \overline{\widehat{w}}(\zeta) - 1 \leq \overline{\widehat{w}}^*(\zeta) \leq \overline{\widehat{w}}(\zeta), \quad \underline{\widehat{w}}(\zeta) - 1 \leq \underline{\widehat{w}}^*(\zeta) \leq \underline{\widehat{w}}(\zeta)$$

again by (17).

Theorem 3.5. *Let ζ be a transcendental real number. Then the above defined exponents satisfy*

$$(35) \quad \frac{(\overline{\widehat{w}}(\zeta) + 1)^2}{4\overline{\widehat{w}}(\zeta)} \leq \overline{\widehat{\lambda}}(\zeta) \leq 1 + \frac{1}{\underline{\widehat{w}}(\zeta)}, \quad \frac{(\underline{\widehat{w}}(\zeta) + 1)^2}{4\underline{\widehat{w}}(\zeta)} \leq \underline{\widehat{\lambda}}(\zeta) \leq 1 + \frac{1}{\overline{\widehat{w}}(\zeta)}.$$

and

$$(36) \quad \overline{\widehat{w}}(\zeta) \leq \min \left\{ 2, \overline{w}(\zeta), 1 + \frac{4}{\underline{w}(\zeta)} \right\}, \quad \underline{\widehat{w}}(\zeta) \leq \min \left\{ 2, \underline{w}(\zeta), 1 + \frac{4}{\overline{w}(\zeta)} \right\}.$$

The particular consequence that $\overline{w}(\zeta) = \infty$ implies $\underline{\widehat{w}}(\zeta) = 1$ was already noticed in [15, Corollary 2.5]. From (35) we can also deduce that $\overline{\widehat{w}}(\zeta) > 1$ implies $\overline{\widehat{\lambda}}(\zeta) > 1$, and $\underline{\widehat{w}}(\zeta) > 1$ implies $\underline{\widehat{\lambda}}(\zeta) > 1$. We believe that the converse implications hold as well. German [19] established refinements of the transference principle (9) for the uniform exponents, however they are again insufficient for such problems. It would be nice to include the exponents on $\overline{w}^*(\zeta)$ and $\underline{w}^*(\zeta)$ in the picture, related to the Wirsing problem. However, we do not know what to conjecture. We remark that from the sparse present results on the exponents $\widehat{w}_n, \widehat{\lambda}_n$, we cannot exclude that the quantities in (32), (33) all equal 1 for any transcendental real number ζ .

4. APPLICATIONS: METRIC THEORY AND SPECTRA

4.1. The joint spectrum of $(\lambda_n)_{n \geq 1}$. We study the set of sequences $\{(\lambda_n(\zeta))_{n \geq 1} : \zeta \in \mathbb{R}\}$, which we will refer to as the *joint spectrum of $(\lambda_n(\zeta))_{n \geq 1}$* . Bugeaud [11, Theorem 4] showed the existence of transcendental real ζ such that $\lambda_n(\zeta) = 1$ for all $n \geq 1$. Thus the constant 1 sequence belongs to the joint spectrum. Moreover [11, Theorem 5] asserted that for given $\lambda \in [1, 3]$ there exists transcendental real ζ such that $\lambda_1(\zeta) = \lambda$ and $\lambda_2(\zeta) = 1$. Both claims are sharp in some sense. Indeed, in view of

$$(37) \quad \lambda_{nk}(\zeta) \geq \frac{\lambda_k(\zeta) - n + 1}{n}, \quad k \geq 1, n \geq 1,$$

from [11, Lemma 1], for $k = 2, n = 1$ we see that $\lambda_1(\zeta) \leq 3$ when $\lambda_2(\zeta) = 1$. A conjectured generalization of (37) was formulated in [28]. As shown by the author [28], there is equality in (37) if $\lambda_{nk}(\zeta) > 1$, in particular

$$(38) \quad \lambda_n(\zeta) = \frac{\lambda_1(\zeta) - n + 1}{n}, \quad \text{if } \lambda_n(\zeta) > 1.$$

Hence we cannot have $\lambda_n(\zeta) > 1$ for all $n \geq 1$, unless ζ is a Liouville number, that is $\lambda_1(\zeta) = \infty$, and in this case the joint spectrum is the constant ∞ sequence by (37) as observed in [11, Corollary 2]. The identity (38) implied a negative answer on Bugeaud's [11, Problem 2] where he proposed that the conditions (7), (8) and (37) might be the only limitations for the joint spectrum of $(\lambda_n(\zeta))_{n \geq 1}$. Our Theorem 2.1 and Corollary 2.2 clearly again show that this is far from being true.

As a consequence of Theorem 3.3, we determine the joint spectrum of $(\lambda_n(\zeta))_{n \geq 1}$ among U_2 -numbers ζ , thereby among all ζ satisfying $\lambda_n(\zeta) > 1/2$ for all $n \geq 1$ by Theorem 2.1.

Theorem 4.1. *Let ζ be a U_2 -number with $w_1(\zeta) = w \in [1, \infty)$. Then*

$$(39) \quad \lambda_n(\zeta) = \frac{w + 1 - n}{n}, \quad 1 \leq n \leq \frac{w + 1}{2},$$

$$(40) \quad \lambda_n(\zeta) = \lambda_{n,2}(\zeta) = 1, \quad n \geq \frac{w + 1}{2}.$$

In particular if $w = 1$ then $\lambda_n(\zeta) = 1$ for all $n \geq 1$. The sequences of the form

$$(41) \quad \left(w, \frac{w-1}{2}, \frac{w-2}{3}, \dots, \frac{w+1 - \lfloor \frac{w+1}{2} \rfloor}{\lfloor \frac{w+1}{2} \rfloor}, 1, 1, 1, \dots \right), \quad w \geq 1,$$

coincide precisely with the sequences $(\lambda_n(\zeta))_{n \geq 1}$ induced by the set of U_2 -numbers ζ . In particular they all belong to the joint spectrum of $(\lambda_n)_{n \geq 1}$. Conversely, the sequences in (41) with $w \in [1, \infty]$ are precisely those sequences in the joint spectrum of $(\lambda_n)_{n \geq 1}$ with $\lambda_n(\zeta) > \frac{1}{2}$ for all $n \geq 1$.

The claims vastly generalize both [11, Theorem 4 and Theorem 5] mentioned above. In the proof we will use the existence of U_2 -numbers with any prescribed value $w_1(\zeta) \in [1, \infty)$. We point out that more generally Alniaçik [2] essentially constructed U_n -numbers ζ with prescribed value of $w_1(\zeta) \in [1, \infty)$, for any $n \geq 2$ (although he only explicitly stated the case $w_1(\zeta) = 1$ in [2]). See also [3] for U -numbers with small transcendence degree. However, the existence of U_n -numbers which satisfy the hypothesis $w_{n-1}(\zeta) = n - 1$,

let alone the more general hypothesis, in Theorem 3.3 is open for $n \geq 3$, which among other things prevents us from generalizing Theorem 4.1. Observe that even the strong hypothesis for (29) would be covered by an affirmative answer on Problem 1.

4.2. Upper bounds for $\widehat{\lambda}_n(\zeta)$. Assume $n \geq 1$ is an integer and ζ a transcendental real number with the property $w_n(\zeta) < n + 1$. If we let $m = 2n$, as a consequence of (24) we obtain

$$(42) \quad \lambda_{2n}(\zeta) \leq \frac{1}{\widehat{w}_n(\zeta)} \leq \frac{1}{n}.$$

Upon the assumption $w_n(\zeta) < n + 1$, the classical estimates (8) and (9) would only yield $\lambda_{2n}(\zeta) \leq \lambda_n(\zeta) < \frac{2}{n}$, so (42) yields an improvement by the factor 2. We can use the conditional result (42) to sharpen the best known upper bound for the exponent $\widehat{\lambda}_n(\zeta)$ for even n . The problem on determining such bounds dates back to Davenport and Schmidt [17] who studied it due to its connection to the Wirsing Problem [38]. Their original result has been refined for odd n by Laurent [22] and slightly for even n recently in [29, Theorem 2.3]. Our new refinement for even n is again based on [29, Theorem 2.1]. It asserts that for m, n positive integers and ζ any transcendental real number the estimate

$$(43) \quad \widehat{\lambda}_{m+n-1}(\zeta) \leq \max \left\{ \frac{1}{w_m(\zeta)}, \frac{1}{\widehat{w}_n(\zeta)} \right\}$$

holds. The addition of (42) leads to a significantly better bound.

Theorem 4.2. *Let $n \geq 1$ be an integer and ζ a transcendental real number. Then we have*

$$(44) \quad \widehat{\lambda}_{2n}(\zeta) \leq \sqrt{\left(n + \frac{1}{2n}\right)^2 - \frac{1}{n}} - n + \frac{1}{2n}.$$

In case of $\lambda_{2n}(\zeta) > \frac{1}{n}$, the stronger bound $\widehat{\lambda}_{2n}(\zeta) \leq \frac{1}{n+1}$ holds.

Proof. The estimate (43) for $m = n + 1$ reads

$$\widehat{\lambda}_{2n}(\zeta) \leq \max \left\{ \frac{1}{w_n(\zeta)}, \frac{1}{\widehat{w}_{n+1}(\zeta)} \right\}.$$

In case of $w_n(\zeta) \geq n + 1$, by (3) we infer $\widehat{\lambda}_{2n}(\zeta) \leq (n + 1)^{-1}$, which is smaller than the right hand side in (44). In case of $w_n(\zeta) < n + 1$, we may apply (42). We insert this value $\lambda_{2n}(\zeta) = n^{-1}$ in the reformulation

$$\widehat{\lambda}_{2n}(\zeta) \leq -\frac{2n - 2 + (2n - 1)\lambda_{2n}(\zeta)}{2} + \sqrt{\left(\frac{2n - 2 + (2n - 1)\lambda_{2n}(\zeta)}{2}\right)^2 + (2n - 1)\lambda_{2n}(\zeta)}$$

of Schmidt and Summerer [35, (1.21)], and elementary rearrangements lead to (44). Reversing the proof we see that $\widehat{\lambda}_{2n}(\zeta) > \frac{1}{n}$ implies $w_n(\zeta) \geq n + 1$, and as above we infer the bound $\widehat{\lambda}_{2n}(\zeta) \leq (n + 1)^{-1}$. \square

The bound in (44) is of asymptotic order $\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3})$. We obtain a reasonable improvement to the old bound $\widehat{\lambda}_{2n}(\zeta)$ of order $\frac{1}{n} - \frac{1}{2n^3} + O(n^{-4})$ in [29, Theorem 2.3]. For $n = 1$ and $n = 2$ we compute

$$\widehat{\lambda}_2(\zeta) \leq \frac{\sqrt{5}-1}{2} = 0.6180\dots, \quad \widehat{\lambda}_4(\zeta) \leq \frac{\sqrt{73}-7}{4} = 0.3860\dots$$

The bound for $\widehat{\lambda}_2(\zeta)$ is well-known to be sharp as shown by Roy [24]. Roy [26] also established the bound $\widehat{\lambda}_3(\zeta) \leq (2 + \sqrt{5} - \sqrt{7 + 2\sqrt{5}})/2 = 0.4245\dots$, which previously represented the best known bound for $\widehat{\lambda}_4(\zeta)$ as well. Our new bound for $\widehat{\lambda}_4(\zeta)$ is finally smaller. Improvements of (44) can be made for $n \geq 2$, conditioned on the conjecture of Schmidt and Summerer proposed in [36, page 92] concerning the minimum value of the quotient $\lambda_n(\zeta)/\widehat{\lambda}_n(\zeta)$ in terms of $\widehat{\lambda}_n(\zeta)$. Concretely, applying [31, (30)] we obtain as an conditional upper bound for $\widehat{\lambda}_{2n}(\zeta)$ the implicit solution λ of the polynomial equation

$$n^{2n}\lambda^{2n+1} - (n+1)\lambda + 1 = 0,$$

in the interval $(\frac{1}{n+1}, \frac{1}{n})$. The resulting conditioned numerical bounds for some small n can be computed as

$$\widehat{\lambda}_4(\zeta) \leq 0.3706\dots, \quad \widehat{\lambda}_6(\zeta) \leq 0.2681\dots, \quad \widehat{\lambda}_{20}(\zeta) \leq 0.0928\dots$$

They turn out to be closer to $\frac{1}{n+1}$ than to $\frac{1}{n}$ for large n . In comparison, the unconditional bounds from (44) are numerically given by $\widehat{\lambda}_6(\zeta) \leq 0.2803\dots$ and $\widehat{\lambda}_{20}(\zeta) \leq 0.0950\dots$

4.3. On the case $\widehat{w}_n(\zeta) > n$. Using Schmidt's Subspace Theorem, Adamczewski and Bugeaud [1] found explicit upper bounds for the exponents $w_m(\zeta)$ for numbers ζ that admit many very small polynomial evaluations $|P_i(\zeta)|$ at $P_i \in \mathbb{Z}[T]$ of bounded degree and high rate in the sense of $\log H(P_{i+1})/\log H(P_i)$ being absolutely bounded. See also A. Baker [4] for earlier results in the case $n = 1$. In [1, Section 5] they provided types of numbers that fall into this category. Any number that satisfies

$$(45) \quad \widehat{w}_n(\zeta) > n, \quad \text{for some } n \geq 2,$$

and is not U_m -numbers for some $m \leq n$, has the desired property. In fact we can exclude the case $m = n$ by [15, Corollary 2.5], and $m = 1$ as well by [28, Theorem 1.12]. So there is no additional condition when $n = 2$. For such numbers they established the exponential bound

$$(46) \quad w_m^*(\zeta) \leq \exp(c \cdot (\log 3m)^n (\log \log 3m)^n), \quad m \geq n+1,$$

where $c = c(\zeta) > 0$ is some ineffective constant [1, Theorem 4.2, 5.3]. In particular numbers that satisfy (45) cannot be U_m -numbers for $m > n$. If we replace (45) by the (at least formally) stronger condition

$$(47) \quad \widehat{w}_n^*(\zeta) > n, \quad \text{for some } n \geq 2,$$

the same conclusion (46) holds without any additional condition by [15, Theorem 2.4]. It is probable that the condition (45) in fact implies $\widehat{w}_n(\zeta) = \widehat{w}_n^*(\zeta)$, Bugeaud [12, Problem 9.14] recently posed the case $n = 2$ as a problem. Another class of ζ satisfying the property are Champerowne-type numbers. These are the numbers with expansion in

some base $b \geq 2$ of the form $\zeta = \zeta_{b,P} = 0.(P(1))_b(P(2))_b \dots$, where $(P(h))_b$ is the integer h written in base b , and $P \in \mathbb{Z}[T]$ non-constant. The classical Champerowne number is obtained for $b = 10$ and $P(T) = T$. We have

$$(48) \quad w_m^*(\zeta_{b,P}) \leq (2m)^{c' \cdot \log \log 3m}, \quad m \geq 1,$$

where $c' = c'(\zeta_{b,P})$ is again a suitable constant [1, Theorem 3.1, 5.1]. See [1, Section 5] for more examples. Corollary 3.2 combined with (46) and (48) yields an estimate for the minimum decay of the exponents $\lambda_N(\zeta)$ for large N , in the case of Champerowne-type numbers by roughly some (ineffective) negative power of N .

Corollary 4.3. *Let ζ be a real number and $\varepsilon > 0$ arbitrarily small. First assume that either (45) and if $n \geq 3$ additionally $w_{n-1}(\zeta) < \infty$ holds, or (47) holds. Then there exists a constant $d = d(\zeta, \varepsilon) > 0$ so that*

$$(49) \quad \lambda_N(\zeta) \leq \exp(-d(\log N)^{\frac{1-\varepsilon}{n}}), \quad N \geq 1.$$

For $\zeta = \zeta_{b,P}$ any Champerowne-type number we have the stronger decay

$$(50) \quad \lambda_N(\zeta) \leq \exp(-d'(\log N)^{1-\varepsilon}), \quad N \geq 1,$$

for suitable $d' = d'(\zeta, \varepsilon) > 0$. In particular in both cases $\lim_{N \rightarrow \infty} \lambda_N(\zeta) = 0$.

Proof. We show (49). It follows from the preceding comments that the condition for (46) is satisfied in any of the given cases. Combination with (16) and some crude estimates imply $w_m(\zeta) \leq Y := \exp(c'(\log m)^{n(1+\varepsilon)})$ with some $c' = c'(\zeta, \varepsilon)$ possibly slightly larger than c . If we let $N = \lceil Y \rceil + 2m$, from (25) we obtain $\lambda_N(\zeta) \leq 1/m$. Hence, for large N and suitable d , the claim (49) follows from elementary estimates and rearrangements. Since ζ is not a U_1 -number and thus $\lambda_N(\zeta) \leq \lambda_1(\zeta) < \infty$ for all $N \geq 1$, by increasing d if necessary we may consider any $N \geq 1$. The claim (50) for Champerowne-type numbers follows in a similar way from (48) and Corollary 3.2. \square

Upon very similar assumptions as for (46), for considerably small integers m significantly smaller upper bounds for $w_m(\zeta)$ were established in [15]. Indeed [15, Theorem 2.4] can be reformulated in the following way. Upon the assumption $\widehat{w}_n^*(\zeta) - (n + u - 1) > 0$ for some integer $u \geq 1$, we have

$$(51) \quad w_{n+j}(\zeta) \leq \frac{(n-1)\widehat{w}_n^*(\zeta)}{\widehat{w}_n^*(\zeta) - (n+j)}, \quad 0 \leq j \leq u-1.$$

Similarly, if $\widehat{w}_n(\zeta) - (n + u - 1) > 0$ and additionally

$$(52) \quad w_{n+u-1}(\zeta) > w_{n-1}(\zeta), \quad \text{or} \quad \widehat{w}_n(\zeta) = \widehat{w}_n^*(\zeta)$$

holds, then [15, Theorem 2.2] analogously asserts

$$(53) \quad w_{n+j}(\zeta) \leq \frac{(n-1)\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta) - (n+j)}, \quad 0 \leq j \leq u-1.$$

In particular $w_{n+u-2}(\zeta) < (n-1)(2n-1) = 2n^2 - 3n + 1$ if $u \geq 2$ by (17). The estimate for $w_{n+u-1}(\zeta)$ is still reasonably good unless $\widehat{w}_n^*(\zeta)$ (or $\widehat{w}_n(\zeta)$) is very close to $n + u - 1$. As another new contribution, by combining results from Section 3 we infer upper bounds for

the next larger exponent $w_{n+u}(\zeta)$. They turn out to be better than (46) for $m = n + u$, upon a stronger assumption. We start with the most general version and specify below.

Theorem 4.4. *Let ζ be a transcendental real number and $n \geq 2$ and $u \geq 1$ be integers. If $\widehat{w}_n^*(\zeta) > n + u - 1$ is satisfied, then we have*

$$(54) \quad w_{n+u}(\zeta) \leq \frac{\widehat{w}_n^*(\zeta)^3 - u\widehat{w}_n^*(\zeta)^2 + (nu + n - 1)\widehat{w}_n^*(\zeta) - n^2}{(\widehat{w}_n^*(\zeta) - n)(\widehat{w}_n^*(\zeta) - n - u + 1)}.$$

If we assume $\widehat{w}_n(\zeta) > n + u - 1$ and additionally (52) holds, then we have

$$(55) \quad w_{n+u}(\zeta) \leq \frac{\widehat{w}_n(\zeta)^3 - u\widehat{w}_n(\zeta)^2 + (nu + n - 1)\widehat{w}_n(\zeta) - n^2}{(\widehat{w}_n(\zeta) - n)(\widehat{w}_n(\zeta) - n - u + 1)}.$$

In particular, if we have either $\widehat{w}_n^(\zeta) - (n + u - 1) =: \delta_1 > 0$, or $\widehat{w}_n(\zeta) - (n + u - 1) =: \delta_2 > 0$ and (52), then for some effectively computable constant $c > 0$ we have*

$$(56) \quad w_{n+u}(\zeta) \leq c \cdot \frac{n^3}{\delta_i^2},$$

for $i = 1$ and $i = 2$ respectively. In case of $u \geq 2$ we have $w_{n+1}(\zeta) \leq c'n^3/\delta_i$.

We highlight the case $u = 1$, where the involved conditions become more natural and the resulting bounds can be rearranged to a nicer form as well.

Corollary 4.5. *Let ζ be a transcendental real number and assume (47) holds for some integer $n \geq 2$. Then we have*

$$(57) \quad w_{n+1}(\zeta) \leq \frac{\widehat{w}_n^*(\zeta)^3 - \widehat{w}_n^*(\zeta)}{(\widehat{w}_n^*(\zeta) - n)^2} - 1.$$

If we assume the weaker condition (45) and additionally (52) for $u = 1$, then similarly

$$(58) \quad w_{n+1}(\zeta) \leq \frac{\widehat{w}_n(\zeta)^3 - \widehat{w}_n(\zeta)}{(\widehat{w}_n(\zeta) - n)^2} - 1.$$

In particular, if either $\widehat{w}_n^(\zeta) - n =: \delta_1 > 0$ holds or $\widehat{w}_n(\zeta) - n =: \delta_2 > 0$ and (52) holds, then we have $w_{n+1}(\zeta) < 8n^3/\delta_i^2$ for $i = 1$ and $i = 2$ respectively.*

Proof. Let $u = 1$ in Theorem 4.4 and rearrange the right hand side to obtain (57) and (58). The factor $8n^3$ reflects a crude estimate of the nominator using (17). \square

We expect that the unpleasant condition $w_n(\zeta) > w_{n-1}(\zeta)$ for (58) can be dropped, which would yield an unconditioned improvement of (46) for $m = n + 1$. We can prove this to be true in the case $n = 2$.

Corollary 4.6. *Let ζ be a real number that satisfies $\widehat{w}_2(\zeta) > 2$. Then we have*

$$w_3(\zeta) \leq \frac{\widehat{w}_2(\zeta)^3 - \widehat{w}_2(\zeta)^2 + 3\widehat{w}_2(\zeta) - 4}{(\widehat{w}_2(\zeta) - 2)^2} \leq \frac{d}{(\widehat{w}_2(\zeta) - 2)^2},$$

where we may choose $d = 14.9444$.

Proof. Application of (58) for $n = 2$ yields the left inequality. We have to check that $\widehat{w}_2(\zeta) > 2$ implies its condition $w_1(\zeta) < w_2(\zeta)$. In [15, Theorem 2.5] it was shown that $\min\{w_m(\zeta), \widehat{w}_n(\zeta)\} \leq m + n - 1$ holds for any positive integers m, n and transcendental real ζ . Indeed, with $m = 1$ and $n = 2$, since $\widehat{w}_2(\zeta) > 2$ this is only possible if $w_1(\zeta) \leq 2 < \widehat{w}_2(\zeta) \leq w_2(\zeta)$. Finally the numeric constant can be derived from $\widehat{w}_2(\zeta) \leq (3 + \sqrt{5})/2$, see Davenport and Schmidt [17]. \square

A special subclass of numbers with the property $\widehat{w}_2(\zeta) > 2$ are Sturmian continued fractions, see [14]. For such numbers Corollary 4.3 and Corollary 4.6 apply. Previously it was not even known that for such numbers $\lim_{N \rightarrow \infty} \lambda_N(\zeta) = 0$ holds. On the other hand, the exponent $w_3(\zeta)$ for Sturmian continued fractions has been explicitly determined, from [14] and [30, Theorem 2.1] we know that $w_2(\zeta) = w_3(\zeta) = \widehat{w}_2(\zeta)/(\widehat{w}_2(\zeta) - 2)$. However, there are many more numbers that satisfy $\widehat{w}_2(\zeta) > 2$, see for example Roy [25], and in general previously we had no explicit upper bounds for $w_3(\zeta)$ in terms of $\widehat{w}_2(\zeta)$. No number with the property (45) for some $n \geq 3$ is known.

4.4. Metric theory. Now we turn to the metric problem of determining the Hausdorff dimensions

$$h_N^\lambda = \dim(H_N^\lambda), \quad H_N^\lambda := \{\zeta \in \mathbb{R} : \lambda_N(\zeta) \geq \lambda\}$$

posed in [11, Problem 2]. We use the subscript index N instead of n to avoid confusion in the proofs later. Obviously the values h_N^λ decay in both variables N, λ . Furthermore, $0 \leq h_N^\lambda \leq 1$ for all N and λ , and $h_N^\lambda = 1$ when $\lambda \leq 1/N$ by (7). Usually one is interested in the values h_N^λ as a function of $\lambda \in [1/N, \infty]$ for fixed N . For parameters greater than one, as a consequence of (38) and the one-dimensional formula by Jarník [20] it was shown in [28, Corollary 1.8] that

$$(59) \quad h_N^\lambda = \frac{2}{(1 + \lambda)N}, \quad N \geq 1, \lambda > 1.$$

For $N = 2$, the problem is solved for parameters $\lambda \leq 1$ as well, Beresnevich, Dickinson and Velani [8] showed that

$$(60) \quad h_2^\lambda = \frac{2 - \lambda}{1 + \lambda}, \quad \frac{1}{2} \leq \lambda \leq 1.$$

For $N \geq 3$ and $\lambda \leq 1$, the problem of determining h_N^λ is open. The lower bound

$$(61) \quad h_N^\lambda \geq \frac{2}{(1 + \lambda)N}, \quad N \geq 1, \lambda \geq \frac{1}{N}$$

follows from (37) as noticed in [11]. For λ close to $1/N$, Beresnevich [7] showed

$$(62) \quad h_N^\lambda \geq \frac{N + 1}{1 + \lambda} - (N - 1), \quad \frac{1}{N} \leq \lambda \leq \frac{3}{2N - 1}.$$

He expects equality to hold in the given interval. The upper bounds in (60) clearly hold for $N \geq 3$ and $\lambda \in [1/2, 1]$ as well, however no improvement has been established yet and for $\lambda < 1/2$ in fact nothing is known. For technical reasons we also introduce the auxiliary variations

$$g_N^\lambda = \dim(G_N^\lambda), \quad G_N^\lambda := \{\zeta \in \mathbb{R} : \lambda_N(\zeta) > \lambda\},$$

of H_N^λ, h_N^λ . Clearly $h_N^{\lambda+\epsilon} \leq g_N^\lambda \leq h_N^\lambda$ for any N, λ and $\epsilon > 0$. We in fact expect $g_N^\lambda = h_N^\lambda$ for all N, λ , but this seems not to be completely obvious.

Essentially by (13) and Bernik's formula (5), we can refine the lower bounds in (61) and establish non-trivial upper bounds. We formulate several variants of new results. First, comparable to (62), we estimate h_N^λ for parameters λ in a fixed ratio with the trivial lower bound $1/N$, asymptotically for large N .

Theorem 4.7. *Let $\tilde{\lambda} \geq 1$ be a parameter, and for $N \geq 1$ let $\theta_N = \tilde{\lambda} \cdot \frac{1}{N}$. Then we have*

$$(63) \quad h_N^{\theta_N} \geq \frac{1}{2\tilde{\lambda} - 1 + 2\sqrt{(\tilde{\lambda})^2 - \tilde{\lambda}}} - O(N^{-1}) \geq \frac{1}{4\tilde{\lambda}} - O(N^{-1}), \quad N \geq 1, \tilde{\lambda} \geq 1.$$

On the other hand, we have

$$(64) \quad h_N^{\theta_N} \leq \frac{1}{\tilde{\lambda} - 2} + O(N^{-1}), \quad N \geq 1, \tilde{\lambda} \geq 3.$$

Furthermore

$$(65) \quad h_N^{\theta_N} \leq \frac{2\tilde{\lambda}}{2\tilde{\lambda} - 1 + \sqrt{4\tilde{\lambda}^2 - 8\tilde{\lambda} + 1}} + O(N^{-1}), \quad N \geq 1, \tilde{\lambda} \geq 2,$$

and for even N moreover

$$(66) \quad g_N^{\theta_N} \leq \frac{N+2}{N+4}, \quad \tilde{\lambda} = 2, N \in \{2, 4, 6, \dots\}.$$

Observe that (61) would only lead to a lower bound of decay $O(N^{-1})$, instead of the absolute lower bound in (63). Khintchine's transference principle (9) combined with (5) admits no conclusion concerning lower bounds for $h_N^{\theta_N}$ for large N , and only yields an upper bound of the form $1 - O(N^{-1})$, reasonably weaker than (64) for $\tilde{\lambda} > 2$, and for $\tilde{\lambda} = 2$ slightly weaker than (66). In both cases the implied constants depend on $\tilde{\lambda}$ only. For small parameters $\tilde{\lambda} \in [1, \frac{3}{2}]$, one readily checks that the bound (63) is weaker than Beresnevich's bound (62) as expected, unless for $\tilde{\lambda} = 1$ when both equal 1. The bound in (65) is stronger than (64) for parameters roughly in the interval $\tilde{\lambda} \in (2, 3.5321\dots)$. The bounds in (65) are larger than $\frac{1}{2}$ for any $\tilde{\lambda} \geq 2$, whereas from (62) we expect $h_N^{\theta_N} = \frac{1}{2}$ for $\theta_N = 3/(2N-1)$, which corresponds to a parameter $\tilde{\lambda} < \frac{3}{2}$ in the notation of Theorem 4.7. However, for $\tilde{\lambda} > 1.8 + o(1)$ as $N \rightarrow \infty$, the bound in (64) is larger than the dimension formula in (62) extended to the right. Thus for larger parameters (62) can no longer represent the dimension formula for h_N^λ , as predicted for roughly $\tilde{\lambda} \leq \frac{3}{2}$.

Now we investigate the case of fixed $\lambda > 0$, and again aim to derive asymptotic bounds for h_N^λ as the dimension N grows.

Theorem 4.8. *Let $\lambda \in (0, 1]$ be given. Then we have*

$$(67) \quad g_N^\lambda \leq \frac{[\lambda^{-1}] + 1}{N - 2[\lambda^{-1}] + 2} < \frac{\lambda^{-1} + 2}{N}, \quad N \geq 3[\lambda^{-1}] - 1.$$

Conversely, we have

$$(68) \quad h_N^\lambda \geq \frac{(1+K)(1+\lambda-K\lambda)}{(N+1-K)(1+\lambda)}, \quad K = \left\lfloor \frac{1+\lambda}{2\lambda} \right\rfloor, \quad N \geq \lfloor \lambda^{-1} \rfloor.$$

In particular, there exist positive constants $c_1(\lambda), c_2(\lambda)$ such that

$$\frac{c_1(\lambda)}{N} \leq h_N^\lambda \leq \frac{c_2(\lambda)}{N}, \quad N \geq 1.$$

The bound in (67) is good when n is large compared to λ^{-1} . For parameters $\lambda \in (\frac{1}{3}, 1]$, the bound in (68) coincides with (61), for $\lambda \leq \frac{1}{3}$ (hence $N \geq 3$) it provides a strict improvement. It is discontinuous for λ a reciprocal of an odd positive integer. For fixed N , a refined treatment leads to a slightly better continuous bound. Moreover, similar to (65) we can derive effective piecewise constant upper bounds for fixed N .

Theorem 4.9. *Let $N \geq 2$ be an integer. Define the intervals $I_1 = [\frac{N+2}{3N}, \infty)$ and*

$$I_n = \left[\frac{N+2}{N+2Nn+n-n^2}, \frac{N+2}{N+2(n-1)N+(n-1)-(n-1)^2} \right), \quad 2 \leq n \leq N.$$

Then I_1, \dots, I_N form a partition of $[\frac{1}{N}, \infty)$ and we have

$$(69) \quad h_N^\lambda \geq \frac{(1+n)(1+\lambda-n\lambda)}{(1+\lambda)(N+1-n)}, \quad \lambda \in I_n.$$

The bound coincides with (61) for $\lambda \in I_1$ and is strictly larger if otherwise $\lambda \in [\frac{1}{N}, \frac{1}{3} + \frac{2}{3N})$.

Conversely, for $N/n \in (2, 3)$ we have

$$(70) \quad g_N^\lambda \leq \frac{n+1}{N-n+1}, \quad \text{if } \lambda \geq \frac{1}{(1-\gamma)N+(2\gamma-1)n},$$

where

$$\gamma = \frac{N^2 + 2n^2 - 3Nn + 2N - 4n}{(n+1)(N-2n)}.$$

Let $N \geq 4$. One checks that then we have $\frac{1}{3} + \frac{2}{3N} > \frac{3}{2N-1}$. The bound of (62) does not apply in the interval $J_N = (\frac{3}{2N-1}, \frac{1}{3} + \frac{2}{3N})$ and hence (69) provides the best known lower bound for h_N^λ in J_N .

For $N = 11$, the different bounds are illustrated by the Mathematica plots Figure 1 and Figure 2. The red curve depicting our new lower bounds is piecewise a rational function, which coincides with the green curve illustrating (61) for $\lambda \geq 13/33 = 0.3939\dots$, and exceeds it for smaller λ . Beresnevich's bound in blue decays almost linearly in its valid interval. Its continuation would exceed the red curve roughly up to $\lambda = 0.1692\dots$. The piecewise constant gray line depicting upper bounds is derived from (70) with the choice $n = 5$ in the interval $\lambda \in [0.2143, 1/3]$, and from (67) for $\lambda \geq 1/3$, with discontinuities at $\lambda = 0.2143\dots$, $\lambda = 1/3$ and $\lambda = 1/2$. We extended it to the interval $[1/11, 0.2143\dots]$ by the trivial bound 1.

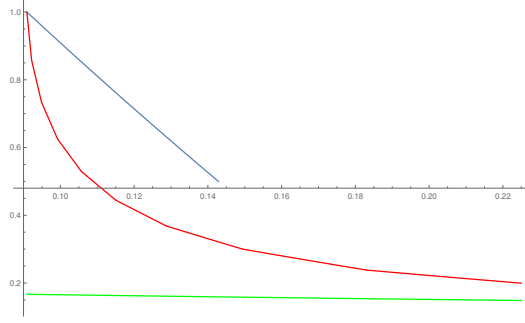


FIGURE 1. Lower bounds for h_{11}^λ : Blue: Beresnevich's lower bound (62) in the valid interval $\lambda \in [\frac{1}{11}, \frac{1}{7}]$. Red: Our lower bound (69) in the sample interval $\lambda \in [\frac{1}{11}, 0.225]$. Green: The lower bound (61) in $\lambda \in [\frac{1}{11}, 0.225]$.

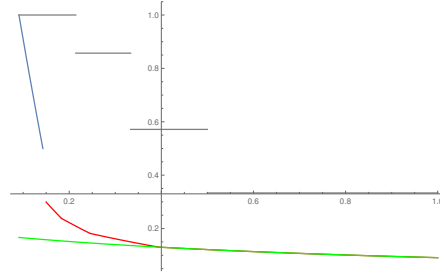


FIGURE 2. Bounds for h_{11}^λ : Blue, red, green as in Figure 1 in the interval $\lambda \in [\frac{1}{11}, 1]$. Gray: Our upper bounds derived from (67) and (70).

Finally we remark that from Theorem 2.5 and (5) we may derive very similar metric results on the dimensions of sets like

$$r_N^w := \dim(R_N^w), \quad R_N^w := \{\zeta \in \mathbb{R} : \widehat{w}_N^*(\zeta) \leq w\} \quad N \geq 1, w \geq 1.$$

We only state the particular consequence that $r_N^{N-\varepsilon} \geq \frac{1}{4} - o(1)$ as $N \rightarrow \infty$, for every $\varepsilon > 0$. For every N , the limit as $\varepsilon \rightarrow 0$ should be one if $w \mapsto r_N^w$ is continuous at $w = N$.

5. PROOFS

5.1. Deduction of the equivalence principles. First we deduce Theorem 2.1 from the partial results in Section 3.

Proof of Theorem 2.1. Theorem 3.3 shows that any U_m -number satisfies

$$\lim_{n \rightarrow \infty} \lambda_n(\zeta) \geq \frac{1}{m-1} > 0.$$

On the other hand in Corollary 3.2 we noticed that otherwise if ζ is no U -number, then $\lim_{n \rightarrow \infty} \lambda_n(\zeta) = 0$. Moreover, when ζ is a U_m -number, then $w_{m-1}(\zeta) < \infty$ and again Corollary 3.2 yields that we actually have $\lambda_n(\zeta) \leq \frac{1}{m-1}$ for large n , so by the above observation there must be equality. In Theorem 3.4 we proved that for T -numbers we have $\limsup_{n \rightarrow \infty} n\lambda_n(\zeta) = \infty$, and $\lim_{n \rightarrow \infty} \lambda_n(\zeta) = 0$ was shown above. Finally the claim for S -numbers was noticed in Corollary 3.2 as well. \square

We now settle the second equivalence principle Theorem 2.4 and Theorem 2.5. Lower bounds for $\widehat{w}^*(\zeta)$ are based on Theorem 2.1 and the relations

$$(71) \quad \widehat{w}_n^*(\zeta) \geq \frac{1}{\lambda_n(\zeta)}, \quad w_n^*(\zeta) \geq \frac{1}{\widehat{\lambda}_n(\zeta)},$$

see [17] and [27]. For upper bounds we employ recent results from [15].

Proof of Theorem 2.4. By (71) and Theorem 2.1, for (18) to hold, ζ must be a U -number. The refined result on U_m -numbers in Theorem 2.1 moreover implies that for ζ a U_m -number we have $\widehat{w}_n^*(\zeta) \geq m-1$ for large n . On the other hand, it was shown in [15, Corollary 2.5] that for any U_m -number we have $\widehat{w}_n^*(\zeta) \leq m$ for all $n \geq 1$. The above implies the left property of (19) for S and T -numbers. We next prove the right claim in (19) for T -numbers. It was shown in [15, Theorem 2.4] that for m, n positive integers the estimate $w_m(\zeta) > m+n-1$ implies

$$(72) \quad \widehat{w}_n^*(\zeta) \leq m + (n-1) \frac{\widehat{w}_n^*(\zeta)}{w_m(\zeta)}.$$

For a T -number ζ and every integer N we have $w_m(\zeta) \geq N^2 m$ for some m . If we choose $n = Nm$, then the condition $w_m(\zeta) > m+n-1$ is satisfied when $N \geq 2$. From (17) and (72) we infer

$$\widehat{w}_{mN}^*(\zeta) \leq m + \frac{2(mN)^2}{N^2 m} \leq 3m.$$

Hence indeed $\widehat{w}_n^*(\zeta)/n = \widehat{w}_{mN}^*(\zeta)/(mN) \leq 3/N$ which tends to 0 as $N \rightarrow \infty$. Finally, for S -number we derive $\widehat{w}_n^*(\zeta) \gg n$ from (71) and Theorem 2.1, the converse follows from above. \square

Remark 6. Besides (71), the inequalities

$$(73) \quad \widehat{w}_n^*(\zeta) \geq \frac{w_n(\zeta)}{w_n(\zeta) - n + 1}, \quad w_n^*(\zeta) \geq \frac{\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta) - n + 1}$$

linking $w_n^*(\zeta)$ and $\widehat{w}_n^*(\zeta)$ with other classical exponents due to Bugeaud and Laurent [14] are known. However, (73) implies $\lim_{n \rightarrow \infty} \widehat{w}_n^*(\zeta) = \infty$ only upon the considerably stronger condition $\liminf_{n \rightarrow \infty} w_n(\zeta)/n = 1$. Similarly, a uniform lower bound for the quantities $\widehat{w}_n^*(\zeta)/n$ would require a uniform upper bound on $w_n(\zeta) - n$ instead of $w_n(\zeta)/n$.

We remark that we can obtain the variants of Theorem 2.4 mentioned in Remark 2 by considering the corresponding variants of (71). Again relation (71) and a refined treatment of the argument for T -numbers leads to a proof of Theorem 2.5.

Proof of Theorem 2.5. The respective left inequalities in (20) and (21) and $\theta(\zeta) \leq \tau(\zeta)^{-1}$ follow immediately from Theorem 2.3 and (71). Concerning the respective right inequalities, the estimates $\overline{w}^*(\zeta) \leq \overline{w}(\zeta) \leq \overline{w}(\zeta)$ and $\widehat{w}^*(\zeta) \leq \widehat{w}(\zeta) \leq \underline{w}(\zeta)$ are an easy consequence of (3) and (17). For the remaining bounds, we refine the argument for T -numbers in the proof of Theorem 2.4. We may assume $\overline{w}(\zeta) > 2$ and $\underline{w}(\zeta) > 2$ respectively, otherwise the left bounds are smaller and the claim is obvious. So assume $\alpha > 2$ is a fixed real number and m is a large integer such that $w_m(\zeta)/m > \alpha$. If n is another integer and we define $\beta = n/m$, then in case of $\beta \leq \alpha - 1$ the condition $w_m(\zeta) > m + n - 1$ of (72) is satisfied. Its application and rearrangements yield

$$\widehat{w}_n^*(\zeta) \leq \frac{\alpha}{\alpha - \beta} m.$$

Dividing by $n = \beta m$ yields

$$\frac{\widehat{w}_n^*(\zeta)}{n} \leq \frac{\alpha}{(\alpha - \beta)\beta}.$$

Let $n = \lfloor m\alpha/2 \rfloor$. Then $\beta = n/m = \alpha/2 + O(1/m)$. Hence, since for $\alpha > 2$ we have $\alpha/2 < \alpha - 1$, the above condition $\beta \leq \alpha - 1$ is satisfied for large m . By inserting we obtain the upper bound $4/\alpha + O(1/m)$ for $\widehat{w}_n^*(\zeta)/n$. By definition we may choose α arbitrarily close to \overline{w} for certain arbitrarily large m , and to \underline{w} for all large m , respectively. The claims (20) and (21) follow. Eventually we show $\theta(\zeta) \leq \tau(\zeta)^{-1}$ to settle (22). Let $\epsilon > 0$ and assume $w_m(\zeta) \geq m^\gamma$. Let $n = m^{\gamma-\epsilon}$ and observe that again the condition $w_m(\zeta) > m + n - 1$ is satisfied for large m . Thus by (72), again for large enough $m \geq m_0(\epsilon)$, we infer

$$\widehat{w}_n^*(\zeta) \leq \frac{m^{\gamma+1}}{(m^\gamma - m^{\gamma-\epsilon} + 1)} \leq 2m = 2n^{1/(\gamma-\epsilon)}.$$

Hence taking logarithms to base n gives $\theta(\zeta) \leq \tau(\zeta)^{-1}$ as γ can be chosen arbitrarily close to $\tau(\zeta)$ and ϵ arbitrarily small. \square

We place the proof of Theorem 2.3 in Section 5.5 as it requires some partial results of the proof of Theorem 3.4.

5.2. Proofs of the upper bounds. Next we show Theorem 3.1. The proof is similar to [29, Theorem 2.1]. Recall the successive minima exponents defined in Section 3.2. As noticed in [27], Mahler's duality can be formulated in the way

$$(74) \quad \lambda_{n,j}(\zeta) = \frac{1}{\widehat{w}_{n,n+2-j}(\zeta)}, \quad \widehat{\lambda}_{n,j}(\zeta) = \frac{1}{w_{n,n+2-j}(\zeta)},$$

with the successive minima exponents defined below Theorem 3.3. Indeed, our proof for the upper bounds for $\lambda_N(\zeta)$ are based on controlling the uniform last successive minimum exponent of the dual problem $\widehat{w}_{N,N+1}(\zeta)$, for suitable N . In fact the dual concept is underlying any proof of upper bounds for $\widehat{\lambda}_n(\zeta)$, where control of $w_{n,n+1}(\zeta)$ leads to bounds for $\widehat{\lambda}_n(\zeta)$. We also recall Gelfond's Lemma, asserting that

$$(75) \quad H(P)H(Q) \ll_n H(PQ) \ll_n H(P)H(Q)$$

holds for any polynomials P, Q each of degree at most n .

Proof of Theorem 3.1. Let n, ζ be as in the theorem and $\epsilon > 0$. By definition of $\widehat{w}_n(\zeta)$, for any large $X \geq X_0(\epsilon)$ there exists an integer polynomial P_X of degree at most n such that

$$H(P_X) \leq X, \quad |P_X(\zeta)| \leq X^{-\widehat{w}_n(\zeta)+\epsilon}.$$

Now choose an integer $k \geq w_n(\zeta)$. The definition of $\widehat{w}_k(\zeta)$ similarly yields an integer polynomial Q_X of degree at most k such that

$$(76) \quad H(Q_X) \leq X, \quad |Q_X(\zeta)| \leq X^{-\widehat{w}_k(\zeta)+\epsilon}.$$

Write $Q_X = R_X S_X$, where R_X consists of the factors dividing P_X as well, and S_X is coprime to P_X . Let $\epsilon > 0$. We claim that, unless R_X is of small height $H(R_X) \ll_\epsilon 1$, we have

$$(77) \quad |R_X(\zeta)| \geq H(R_X)^{-w_n(\zeta)-\epsilon} \gg_{k,\zeta} X^{-w_n(\zeta)-\epsilon},$$

if X was chosen sufficiently large. First notice that the corresponding estimate

$$(78) \quad |U_X(\zeta)| \geq H(U_X)^{-w_n(\zeta)-\epsilon},$$

applies to any irreducible factor U_X of R_X . Indeed, such U_X has degree at most n as it also divides P_X , and by definition of $w_n(\zeta)$ we obtain (78). From (75) we see that this property is essentially (up to a factor depending on k only) preserved when taking arbitrary products, more precisely

$$(79) \quad |R_X(\zeta)| \geq H(R_X)^{-w_n(\zeta)-\epsilon} \gg_{k,\zeta} X^{-w_n(\zeta)-\epsilon}.$$

In case of R_X of small height $H(R_X) \ll_\epsilon 1$, we can even estimate $|R_X(\zeta)| \gg_{n,\zeta} 1$ by the finiteness and since ζ is transcendental. From (76) and (77) we deduce

$$|S_X(\zeta)| = \frac{|Q_X(\zeta)|}{|R_X(\zeta)|} \leq X^{-\widehat{w}_k(\zeta)+w_n(\zeta)+2\epsilon}.$$

Moreover, since S_X divides Q_X , Gelfond's estimate (75) implies $H(S_X) \ll_k H(Q_X) \leq X$. Hence we have

$$(80) \quad \max\{H(P_X), H(S_X)\} \ll_k X, \quad \max\{|P_X(\zeta)|, |S_X(\zeta)|\} \leq X^{-\theta_{k,n}+2\epsilon},$$

with

$$\theta_{k,n} = \min\{\widehat{w}_n(\zeta), \widehat{w}_k(\zeta) - w_n(\zeta)\}.$$

Let $d_X = d \leq n$ be the degree of P_X and $e_X = e \leq k$ be the degree of S_X . Then, since P_X and Q_X are coprime, the set of polynomials

$$\mathcal{P}_X := \{P_X, TP_X, \dots, T^{e-1}P_X, S_X, TS_X, \dots, T^{d-1}S_X\}$$

is linearly independent and spans the space of polynomials of degree at most $d + e - 1 \leq k + n - 1$. In case of strict inequality $d + e - 1 < k + n - 1$ for some X , we consider

$$\mathcal{R}_X = \mathcal{P}_X \cup \{T^d S_X, T^{d+1} S_X, \dots, T^{k+n-1-e} S_X\}$$

instead of \mathcal{P}_X (see also the proof of Proposition 5.1 below). Clearly \mathcal{R}_X is linearly independent as well, and spans the space of polynomial of degree at most $N := k + n - 1$. In any case, in view of (80) and since X was arbitrary and we may choose ϵ arbitrarily small, this means

$$\widehat{w}_{N,N+1}(\zeta) \geq \theta_{k,n} = \min\{\widehat{w}_n(\zeta), \widehat{w}_{N-n+1}(\zeta) - w_n(\zeta)\}.$$

Since $\theta_{k,n} > 0$ by construction, Mahler's relation (74) with $j = 1$ further implies $\lambda_N(\zeta) \leq 1/\theta_{k,n}$. We may choose any integer $k > w_n(\zeta)$, and the choice $k = \lceil w_n(\zeta) \rceil$ yields $N = n + k - 1 = \lceil w_n(\zeta) \rceil + n - 1$. The claim (23) follows.

Now we prove (24). We now choose an integer k with strict inequality $k > w_n(\zeta)$, and again obtain (76) for some Q_X of degree at most k for any $X \geq X_0(\epsilon)$. We proceed as above splitting $Q_X = R_X S_X$. By a very similar argument as above, from (75) we derive that Q_X cannot split solely in irreducible polynomials of degree at most n . Thus it must have an irreducible factor of degree at least $n + 1$, which must divide S_X . Hence $R_X = Q_X/S_X$ has degree at most $k - (n + 1)$. In particular if $w_n(\zeta) < n + 1$, for $k = n + 1$ we infer $S_X = Q_X$ and $R_X \equiv 1$ for all large X . From the definition of w_{k-n-1} for sufficiently large $H(R_X)$ we derive

$$(81) \quad |R_X(\zeta)| \geq H(R_X)^{-w_{k-n-1}(\zeta)-\epsilon} \gg_{k,\zeta} X^{-w_{k-n-1}(\zeta)-\epsilon}.$$

In case of small heights of R_X we use the argument from the proof of (23) again. We infer (80) very similarly as above with $\theta_{k,n}$ replaced by the new expression

$$\tilde{\theta}_{k,n} = \min\{\widehat{w}_n(\zeta), \widehat{w}_k(\zeta) - w_{k-n-1}(\zeta)\}.$$

Let $N = k + n - 1$ again, proceeding as above yields

$$\widehat{w}_{N,N+1}(\zeta) \geq \tilde{\theta}_{k,n} = \min\{\widehat{w}_n(\zeta), \widehat{w}_{N-n+1}(\zeta) - w_{N-2n}(\zeta)\}.$$

We may start with any integer $k > w_n(\zeta)$, or equivalently $k \geq \lfloor w_n(\zeta) \rfloor + 1$, which leads to $N \geq \lfloor w_n(\zeta) \rfloor + n$. The claim (24) follows from (74) again as soon as $\theta_{k,n} > 0$, which we can guarantee for $N \leq 3n$ by construction. The condition $w_n < 2n - 1$ is only required for the set of values N in (24) to be non-empty. \square

5.3. Parametric geometry of numbers. The proofs of Section 3.2 and Section 4.1 can be derived in a convenient way, and in fact surprisingly easy, utilizing the parametric geometry of numbers introduced by Schmidt and Summerer [34]. We recall the fundamental concepts, in a slightly modified form to fit our purposes. In particular we restrict to successive powers of a number. Let $\zeta \in \mathbb{R}$ be given and $Q > 1$ a parameter. For $n \geq 1$ and $1 \leq j \leq n + 1$ define $\psi_{n,j}(Q)$ as the maximum of $\eta \in \mathbb{R}$ such that

$$(82) \quad |x| \leq Q^{1+\eta}, \quad \max_{1 \leq j \leq n} |\zeta^j x - y_j| \leq Q^{-\frac{1}{n}+\eta}$$

has j linearly independent solution vectors $(x, y_1, \dots, y_n) \in \mathbb{Z}^{n+1}$. The functions $\psi_{n,j}(Q)$ can be equivalently defined via a lattice point problem, see [34]. As pointed out in [34] they have the properties

$$(83) \quad -1 \leq \psi_{n,1}(Q) \leq \psi_{n,2}(Q) \leq \dots \leq \psi_{n,n+1}(Q) \leq \frac{1}{n}, \quad Q > 1.$$

Let

$$(84) \quad \underline{\psi}_{n,j} = \liminf_{Q \rightarrow \infty} \psi_{n,j}(Q), \quad \overline{\psi}_{n,j} = \limsup_{Q \rightarrow \infty} \psi_{n,j}(Q).$$

These values all belong to the interval $[-1, 1/n]$ by (83). From Dirichlet's Theorem it follows that $\psi_{n,1}(Q) < 0$ for all $Q > 1$ and hence $\underline{\psi}_{n,1} \leq 0$. Similarly, for $1 \leq j \leq n+1$ define the functions $\psi_{n,j}^*(Q)$ as the supremum of η such that

$$(85) \quad H(P) \leq Q^{\frac{1}{n}+\eta}, \quad |P(\zeta)| \leq Q^{-1+\eta}$$

has j linearly independent integer polynomial solutions P of degree at most n . Again put

$$(86) \quad \underline{\psi}_{n,j}^* = \liminf_{Q \rightarrow \infty} \psi_{n,j}^*(Q), \quad \overline{\psi}_{n,j}^* = \limsup_{Q \rightarrow \infty} \psi_{n,j}^*(Q).$$

We have

$$-\frac{1}{n} \leq \psi_{n,1}^*(Q) \leq \psi_{n,2}^*(Q) \leq \dots \leq \psi_{n,n+1}^*(Q) \leq 1, \quad Q > 1.$$

As pointed out in [34] Mahler's relations (74) are essentially equivalent to

$$(87) \quad \underline{\psi}_{n,j} = -\overline{\psi}_{n,n+2-j}^*, \quad \overline{\psi}_{n,j} = -\underline{\psi}_{n,n+2-j}^*, \quad 1 \leq j \leq n+1.$$

Schmidt and Summerer [35, (1.11)] further established the inequalities

$$(88) \quad j\underline{\psi}_{n,j} + (n+1-j)\overline{\psi}_{n,n+1} \geq 0, \quad j\overline{\psi}_{n,j} + (n+1-j)\underline{\psi}_{n,n+1} \geq 0,$$

for $1 \leq j \leq n+1$. The dual inequalities

$$(89) \quad j\underline{\psi}_{n,j}^* + (n+1-j)\overline{\psi}_{n,n+1}^* \geq 0, \quad j\overline{\psi}_{n,j}^* + (n+1-j)\underline{\psi}_{n,n+1}^* \geq 0,$$

can be obtained very similarly. We point out that in case of equality in (88) and (89) respectively, their proofs in [34] directly imply

$$(90) \quad \underline{\psi}_{n,1} = \underline{\psi}_{n,2} = \dots = \underline{\psi}_{n,j}, \quad \overline{\psi}_{n,j+1} = \overline{\psi}_{n,j+2} = \dots = \overline{\psi}_{n,n+1},$$

and

$$(91) \quad \underline{\psi}_{n,1}^* = \underline{\psi}_{n,2}^* = \dots = \underline{\psi}_{n,j}^*, \quad \overline{\psi}_{n,j+1}^* = \overline{\psi}_{n,j+2}^* = \dots = \overline{\psi}_{n,n+1}^*,$$

respectively. Moreover, by [34, Theorem 1.4] the quantities in (84) and (86) are connected to the exponents $\lambda_{n,j}$, $\widehat{\lambda}_{n,j}$ and $w_{n,j}$, $\widehat{w}_{n,j}$ via the identities

$$(92) \quad (1 + \lambda_{n,j}(\zeta))(1 + \underline{\psi}_{n,j}) = (1 + \widehat{\lambda}_{n,j}(\zeta))(1 + \overline{\psi}_{n,j}) = \frac{n+1}{n}, \quad 1 \leq j \leq n+1,$$

and

$$(93) \quad (1 + w_{n,j}(\zeta))\left(\frac{1}{n} + \underline{\psi}_{n,j}^*\right) = (1 + \widehat{w}_{n,j}(\zeta))\left(\frac{1}{n} + \overline{\psi}_{n,j}^*\right) = \frac{n+1}{n}, \quad 1 \leq j \leq n+1.$$

In fact it was only observed for $j = 1$ in [34], but as remarked in [27] it is true as well for $2 \leq j \leq n + 1$ for the same reason.

5.4. Proofs of the lower bounds. The following easy observation will play an important role in the proofs of lower bounds.

Proposition 5.1. *Let m, n be positive integers and ζ be a real transcendental number. Then*

$$(94) \quad w_{m+n, m+i}(\zeta) \geq w_{n, i}(\zeta), \quad \widehat{w}_{m+n, m+i}(\zeta) \geq \widehat{w}_{n, i}(\zeta), \quad 1 \leq i \leq n + 1.$$

Proof. By definition of $w_{n, i}(\zeta)$, for certain arbitrarily large X there exist linearly independent integer polynomials P_1, \dots, P_i of degree at most n with the properties

$$\max_{1 \leq j \leq i} H(P_j) \leq X, \quad \max_{1 \leq j \leq i} |P_j(\zeta)| \leq H(P_j)^{-w_{n, i}(\zeta) + \epsilon}.$$

Without loss of generality assume the degree of P_1 is maximal among the P_j . For any $m \geq 1$ consider the set of polynomials

$$\mathcal{P}_{m, n, i} = \mathcal{P}_{m, n, i}(X) = \{P_1(T), TP_1(T), T^2P_1(T), \dots, T^mP_1(T), P_2(T), \dots, P_i(T)\}$$

It is not hard to see that $\mathcal{P}_{m, n, i}$ consists of $m + i$ polynomials of degree at most $n + m$, is linearly independent as well, and satisfies

$$\max_{P \in \mathcal{P}_{m, n, i}} H(P) \leq X, \quad \max_{P \in \mathcal{P}_{m, n, i}} |P(\zeta)| \leq \max\{1, |\zeta|^m\} H(P)^{-w_{n, i}(\zeta) + \epsilon}.$$

The left inequalities of (94) follow. The right ones are shown similarly using the definition of $\widehat{w}_{n, i}(\zeta)$ and considering any large X . \square

In fact we only need the case $i = 1$. First we deduce Theorem 3.3 from the proposition.

Proof of Theorem 3.3. By (8) it suffices to prove (27) for $n \geq m$. So let $n = m + k$ with $k \geq 0$. From $w_m(\zeta) = \infty$ and Proposition 5.1 we derive $w_{n, k+1}(\zeta) = \infty$. Together with (93) we infer

$$\underline{\psi}_{n, k+1}^* = -\frac{1}{n}.$$

Hence (89) with $j = k + 1$ and (87) yield

$$(95) \quad \underline{\psi}_{n, 1} = -\overline{\psi}_{n, n+1}^* \leq \frac{k+1}{(n+1) - (k+1)} \underline{\psi}_{n, k+1}^* = -\frac{k+1}{(k+m)m}.$$

Inserting in (92) yields $\lambda_n(\zeta) \geq 1/(m-1)$ as asserted. Now assume $m \geq 2$ and $w_{m-1}(\zeta) = m-1$. We first only show $\lambda_n(\zeta) = 1$ for $n \geq m-1$. The properties $w_{m-1}(\zeta) = m-1$ and $\lambda_{m-1}(\zeta) = 1/(m-1)$ are equivalent, which follows for example from (9). Thus for $n \geq m-1$ we have $1/(m-1) = \lambda_{m-1} \geq \lambda_n \geq 1/(m-1)$ by (27) and (8), and our special case of (28) follows. For the general claim (28), observe that reversing the above process, the identity $\lambda_n(\zeta) = 1/m$ implies equality in the inequality in (95). As observed in (90) and (91) above, this can only happen when $\overline{\psi}_{n, k+2}^* = \overline{\psi}_{n, k+3}^* = \dots = \overline{\psi}_{n, n+1}^*$. Noticing $n+1 - (k+1) = m$, Mahler's relations (87) yield $\underline{\psi}_{n, 1} = \underline{\psi}_{n, 2} = \dots = \underline{\psi}_{n, m}$, and by (93) we infer $\lambda_{n, 1}(\zeta) = \dots = \lambda_{n, m}(\zeta)$. The last claim follows similarly from Khintchine's principle (9). \square

Similar considerations and a well-known existence result lead to a proof of Theorem 4.1.

Proof of Theorem 4.1. From combination of Theorem 3.3 and (38) we deduce (39) and (40), apart from $\lambda_{n,2}(\zeta) = 1$. This refinement is inferred as the general claim of (28) in the proof of Theorem 3.3. Indeed, since we already know $\lambda_n(\zeta) = 1$, the same argument implies $\overline{\psi}_{n,n}^* = \overline{\psi}_{n,n+1}^*$ or equivalently $\underline{\psi}_{n,1} = \underline{\psi}_{n,2}$, and (92) yields $\lambda_{n,2}(\zeta) = \lambda_n(\zeta) = 1$. In order to show that any sequence as in (41) belongs to the joint spectrum, it suffices to notice that U_2 -numbers with any prescribed value $w = w_1(\zeta)$ can be constructed by means of continued fractions as pointed out in [10, paragraph 7.6 on page 158]. Eventually, from Theorem 2.1 (or Theorem 3.1) we know that $\lim_{n \rightarrow \infty} \lambda_n(\zeta) > 1/2$ implies ζ must be a U_1 -number or U_2 -number. In case of a U_2 -number the above applies, for a Liouville number ζ the sequence $(\lambda_n(\zeta))_{n \geq 1}$ is constant $\{\infty\}$ anyway as noticed in [11, Corollary 2]. \square

Theorem 3.4 follows similarly as Theorem 3.3, with slightly more computation involved.

Proof of Theorem 3.4. Let m, n be positive integers and $C \geq 1$ a real number to be chosen later and assume we have $w_n(\zeta) \geq Cn$. Proposition 5.1 yields

$$w_{m+n,m+1}(\zeta) \geq nC.$$

With (93) we obtain

$$\underline{\psi}_{m+n,m+1}^* \leq \frac{m+n+1}{(m+n)(1+nC)} - \frac{1}{m+n} = \frac{m+n(1-C)}{(m+n)(1+nC)}.$$

Hence (89) with $j = m+1$ and (87) imply

$$(96) \quad \underline{\psi}_{m+n,1} = -\overline{\psi}_{m+n,m+n+1}^* \leq \frac{m+1}{n} \underline{\psi}_{m+n,m+1}^* = \frac{m+1}{n} \cdot \frac{m+n(1-C)}{(m+n)(1+nC)}.$$

Application of (92) yields

$$(97) \quad \lambda_{m+n}(\zeta) \geq \frac{m+n+1}{m+n} \cdot \frac{1}{1 + \frac{m+1}{n} \frac{m+n(1-C)}{(m+n)(1+nC)}} - 1 = \frac{Cn-m}{m+n(1+C(n-1))}.$$

Let $m = \lceil Rn \rceil$ with the optimal parameter $R = (C-1)/2$. A short computation shows

$$(98) \quad (m+n)\lambda_{m+n}(\zeta) \geq \left(\frac{C+1}{2}\right)^2 \frac{n}{R+1+(n-1)C} - \epsilon > \frac{(C+1)^2}{4C} - 2\epsilon,$$

for $n \geq n_0(C, \epsilon)$. We infer (30) as we may choose C arbitrarily close to \overline{w} and \underline{w} respectively, for certain arbitrarily large n and all large n , respectively. For T -numbers we may choose arbitrarily large C for certain large n , and the claim (31) follows. \square

We kind of dualize the proof for the uniform exponents.

Proof of Theorem 3.5. We first modify the proof of Theorem 3.4 to show the left inequalities of (35). We apply the uniform inequality of Proposition 5.1 to see that if $\widehat{w}_n(\zeta) \geq Cn$ then for any $m \geq 1$ we have

$$\widehat{w}_{m+n,m+1}(\zeta) \geq nC.$$

With (93) we infer

$$\overline{\psi}_{m+n,m+1}^* \leq \frac{m+n+1}{(m+n)(1+nC)} - \frac{1}{m+n} = \frac{m+n(1-C)}{(m+n)(1+nC)}.$$

Again (89) with $j = m+1$ and (87) yield

$$\overline{\psi}_{m+n,1} = -\underline{\psi}_{m+n,m+n+1}^* \leq \frac{m+1}{n} \overline{\psi}_{m+n,m+1}^* = \frac{m+1}{n} \cdot \frac{m+n(1-C)}{(m+n)(1+nC)}.$$

We apply (92) and obtain

$$\widehat{\lambda}_{m+n}(\zeta) \geq \frac{Cn-m}{m+n(1+C(n-1))}.$$

Again with the parameter choice $m = \lceil n(C-1)/2 \rceil$ we obtain the left inequalities in (35) by multiplication with $m+n$. The right inequalities of (35) are a consequence of (43) as we carry out. Let $\epsilon > 0$. By definition of $\widehat{w}(\zeta)$ there exist arbitrarily large n such that $\widehat{w}_n(\zeta) \geq n(\widehat{w}(\zeta) - \epsilon)$. Provided n was chosen sufficiently large, for $m = \lfloor n(\widehat{w}(\zeta) - \epsilon) \rfloor$, from (43) and $w_m(\zeta) \geq m$ we infer

$$\widehat{\lambda}_{m+n-1}(\zeta) = \widehat{\lambda}_{\lfloor n(\widehat{w}(\zeta) - \epsilon) \rfloor + n - 1}(\zeta) \leq \max \left\{ \frac{1}{m}, \frac{1}{\widehat{w}_n(\zeta)} \right\} \leq \frac{1}{n(\widehat{w}(\zeta) - 2\epsilon)}.$$

Since $m+n-1 \leq n(1 + \widehat{w}(\zeta) - \epsilon)$ we obtain

$$(m+n-1)\widehat{\lambda}_{m+n-1}(\zeta) \leq \frac{1 + \widehat{w}(\zeta) - \epsilon}{\widehat{w}(\zeta) - 2\epsilon}.$$

We conclude $\widehat{\lambda}(\zeta) \leq (\widehat{w}(\zeta) + 1)/\widehat{w}(\zeta)$ as $\epsilon \rightarrow 0$ and $n \rightarrow \infty$. The claim $\widehat{\lambda}(\zeta) \leq (\widehat{w}(\zeta) + 1)/\widehat{w}(\zeta)$ is derived similarly, starting with any large n . Finally (36) follows from (17), (34) and Theorem 2.5. \square

5.5. Proofs of Theorem 4.4 and Theorem 2.3. We now deduce Theorem 4.4 from Theorem 3.4.

Proof of Theorem 4.4. First we show that (52) implies (55). Let $N = \lceil w_n(\zeta) + \widehat{w}_n(\zeta) \rceil + n - 1$ and apply (23). We check that the left bound is larger and hence

$$(99) \quad \lambda_N(\zeta) \leq \frac{1}{\widehat{w}_n(\zeta)}.$$

On the other hand, when we write $N = m + (n+u)$, such that $m = \lceil w_n(\zeta) + \widehat{w}_n(\zeta) \rceil - (u+1)$, from (97) and $N \leq w_n(\zeta) + \widehat{w}_n(\zeta) + n$ we obtain

$$(100) \quad \lambda_N(\zeta) \geq \frac{w_{n+u}(\zeta) - m}{N + (n+u-1)w_{n+u}(\zeta)} \geq \frac{w_{n+u}(\zeta) - m}{w_n(\zeta) + \widehat{w}_n(\zeta) + (n+u-1)w_{n+u}(\zeta) + n}.$$

Since $\widehat{w}_n(\zeta) - (n+u-1) > 0$ by assumption, combination of (99) and (100) yields after elementary rearrangements

$$(101) \quad w_{n+u}(\zeta) \leq \frac{(m+1)\widehat{w}_n(\zeta) + w_n(\zeta) + n}{\widehat{w}_n(\zeta) - (n+u-1)}.$$

By assumption (52) we may apply (53), and $j = 0$ reads

$$(102) \quad w_n(\zeta) \leq \frac{(n-1)\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta) - n}.$$

Plugging this in (101) and estimating $m \leq w_n(\zeta) + \widehat{w}_n(\zeta) - u$, after a short calculation we obtain (55).

For the unconditioned bound, notice that the right hand side in (101) is monotonic decreasing as a function of $\widehat{w}_n(\zeta)$ in the interval $(n+u-1, \infty)$. Thus, by $\widehat{w}_n^*(\zeta) \leq \widehat{w}_n(\zeta)$ from (17), upon the assumption $\widehat{w}_n^*(\zeta) - (n+u-1) > 0$ from (101) we may conclude

$$(103) \quad w_{n+u}(\zeta) \leq \frac{(m+1)\widehat{w}_n^*(\zeta) + w_n(\zeta) + n}{\widehat{w}_n^*(\zeta) - (n+u-1)}.$$

Again since $\widehat{w}_n^*(\zeta) - n \geq \widehat{w}_n^*(\zeta) - (n+u-1) > 0$, we apply the unconditioned estimate (51) for $j = 0$ which reads

$$(104) \quad w_n(\zeta) \leq \frac{(n-1)\widehat{w}_n^*(\zeta)}{\widehat{w}_n^*(\zeta) - n},$$

and the same calculation as above shows (54). Finally, the asymptotic claim (56) is obtained from (17). \square

Finally we put the results together to prove Theorem 2.3.

Proof of Theorem 2.3. The inequalities in (13) have already been noticed in Theorem 3.4 and Theorem 3.1. The inequality $\sigma(\zeta) \leq -1/\tau(\zeta)$ follows easily from (25). The reverse inequality $\sigma(\zeta) \geq -1/\tau(\zeta)$ follows from (97) by taking $m = n^{\tau(\zeta)-\delta}$ and letting $\delta \rightarrow 0$, observing that $\log C/\log n$ is arbitrarily close to $\tau(\zeta) - 1$ for certain large n . \square

5.6. Proofs of the metric results. For the proof of Theorem 4.7 we essentially reverse the proofs of Theorem 3.1 and Theorem 3.4, and taking care of the occurring error terms from rounding to integers in the process. For simplicity let

$$t_n^w = \dim(T_n^w), \quad T_n^w = \{\zeta \in \mathbb{R} : w_n(\zeta) \geq w\}.$$

By (5) we have $t_n^w = (n+1)/(w+1)$ for $n \geq 1, w \geq n$.

Proof of Theorem 4.7. For $\tilde{\lambda} > 1$ as in the theorem let

$$C = 2\tilde{\lambda} - 1 + 2\sqrt{(\tilde{\lambda})^2 - \tilde{\lambda}}, \quad \sigma = \frac{2}{1+C},$$

where $C \geq 1$ is a solution to $(C+1)^2/(4C) = \tilde{\lambda}$. Assume N is large and further let $n = \lceil n^* \rceil$ with $n^* = \sigma N$, such that

$$n = \lceil n^* \rceil = \lceil \sigma N \rceil = \left\lceil \frac{2}{1+C} N \right\rceil.$$

Obviously $0 \leq \sigma N - n < 1$. Assume ζ satisfies $w_n(\zeta) \geq Cn$. We want to show that $\lambda_N(\zeta)$ is then essentially bounded below by θ_N . We proceed as in Theorem 3.4 for our present n . For any integer $m \geq 1$, we obtain

$$(105) \quad (m+n)\lambda_{m+n}(\zeta) \geq \frac{(m+n)(Cn-m)}{m+n(1+C(n-1))}.$$

We choose $m = \lceil m^* \rceil$ with $m^* = N - n^* = n^* \cdot (C-1)/2$. Since

$$(106) \quad 0 \leq m - m^* < 1, \quad 0 \leq n - n^* < 1,$$

we conclude

$$(107) \quad m+n \in \{N, N+1\}.$$

Denote $\Phi(m, n)$ the right hand side in (105) treated as a function in two variables. When we replace m, n by m^*, n^* , a computation shows

$$\Phi(m^*, n^*) = \frac{n^* \left(\frac{C+1}{2}\right)^2}{C(n^* - \frac{1}{2}) + \frac{1}{2} - C}$$

We readily deduce that for some constant $c_0 = c_0(C)$ we have

$$\Phi(m^*, n^*) \geq \frac{(C+1)^2}{4C} - \frac{c_0}{n}.$$

Similarly with (106) we easily verify that for some constant $c_1 = c_1(C)$ we have

$$\Phi(m, n) \geq \Phi(m^*, n^*) - \frac{c_1}{n}.$$

Since $n \gg_C N$ by construction, combination of the two inequalities yields

$$\Phi(m, n) \geq \frac{(C+1)^2}{4C} - \frac{c_2}{n} \geq \frac{(C+1)^2}{4C} - \frac{c_3}{N},$$

with $c_2(C) = c_0(C) + c_1(C)$ and some new constant $c_3 = c_3(C)$. Since $\Phi(m, n)$ was the lower bound in (105) and by (107), for some new constant $c_4 = c_4(C)$ we infer

$$N\lambda_N(\zeta) \geq \frac{N}{N+1}\Phi(m, n) \geq \frac{N}{N+1} \cdot \left(\frac{(C+1)^2}{4C} - \frac{c_3}{N} \right) \geq \frac{(C+1)^2}{4C} - \frac{c_4}{N}.$$

Since $(C+1)^2/(4C) = \tilde{\lambda}$, the argument shows

$$T_n^{Cn} \subseteq H_N^{\frac{\varphi}{N}}, \quad \varphi = \tilde{\lambda} - \frac{c_4}{N}.$$

Thus by (5) we estimate the dimension as

$$h_N^{\frac{\varphi}{N}} \geq t_n^{Cn} = \frac{n+1}{Cn+1} \geq \frac{1}{C}.$$

The claim (63) follows by starting with $\tilde{\lambda} + \epsilon$ instead of $\tilde{\lambda}$ and letting ϵ tend to 0. The upper bound (64) follows similarly from (26), the proof is left to the reader.

We show (65). Let $\tilde{\lambda} \geq 2$ be given and assume $N\lambda_N(\zeta) > \tilde{\lambda}$ for some large N . Let $\beta \in [2, 3), \gamma \geq 1$ to be chosen later. For now for simplicity assume $n = N/\beta$ is an integer. Assume $w_n(\zeta) \leq (\beta - 1)n$. Then since $\beta \in [2, 3)$ we may apply (24) to N, n and obtain

$$(108) \quad \lambda_m(\zeta) \leq \frac{1}{m - n + 1 - w_{m-2n}(\zeta)} \leq \frac{1}{(\beta - 1)n - w_{(\beta-2)n}(\zeta)}$$

Now assume

$$w_{(\beta-2)n}(\zeta) \leq \gamma(\beta - 2)n.$$

Then multiplication of (108) with $N = \beta n$ yields

$$(109) \quad N\lambda_N(\zeta) \leq \frac{\beta}{\beta - 1 - \gamma(\beta - 2)}.$$

In other words, if we have

$$N\lambda_N(\zeta) > \frac{\beta}{\beta - 1 - \gamma(\beta - 2)},$$

then either $w_n(\zeta) > (\beta - 1)n$ or $w_{(\beta-2)n}(\zeta) > \gamma(\beta - 2)n$. Hence

$$(110) \quad \mathcal{D} := \mathcal{D}_{N,\beta,\gamma} := G_N^{\frac{\beta}{N(\beta-1-\gamma(\beta-2))}} = \{\zeta \in \mathbb{R} : N\lambda_N(\zeta) > \frac{\beta}{\beta - 1 - \gamma(\beta - 2)}\}$$

is contained in $\mathcal{A} \cup \mathcal{B} = \mathcal{A}_{N,\beta} \cup \mathcal{B}_{N,\beta,\gamma}$ with

$$(111) \quad \mathcal{A} := T_n^{(\beta-1)n}, \quad \mathcal{B} := T_{(\beta-2)n}^{\gamma(\beta-2)n}.$$

The dimensions of the sets \mathcal{A}, \mathcal{B} can be determined with (5) as

$$\dim(\mathcal{A}) = \frac{n+1}{(\beta-1)n+1} = \frac{1}{\beta-1} + O(n^{-1}), \quad \dim(\mathcal{B}) = \frac{(\beta-2)n+1}{\gamma(\beta-2)n+1} = \frac{1}{\gamma} + O(n^{-1}).$$

Hence, for given $\beta \in [2, 3), \gamma \geq 1$, the set \mathcal{D} in (110) has Hausdorff dimension at most

$$(112) \quad \dim(\mathcal{D}) \leq \dim(\mathcal{A} \cup \mathcal{B}) = \max\{\dim(\mathcal{A}), \dim(\mathcal{B})\} = \max\left\{\frac{1}{\beta-1}, \frac{1}{\gamma}\right\} + O(n^{-1}).$$

Put

$$(113) \quad \gamma = \frac{2\tilde{\lambda} - 1 + \sqrt{4\tilde{\lambda}^2 - 8\tilde{\lambda} + 1}}{2\tilde{\lambda}}, \quad \beta = \gamma + 1.$$

For these choices one checks that the right hand side expression in (110) equals $\tilde{\lambda}$. On the other hand, by construction clearly $(\beta - 1)^{-1} = \gamma^{-1}$ and this value yields the right hand side of (65) as an upper bound in (112). One further readily checks that for any given $\tilde{\lambda} \geq 2$, the initial assumptions $\beta \in [2, 3)$ and $\gamma \geq 1$ for β, γ in (113) are satisfied. We still have to deal with the problem that N/β is no integer in general. However, for given $\beta \in [2, 3)$ if we let $n = \lfloor N/\beta \rfloor$ then $\beta' := N/n = \beta + O(N^{-1})$. One checks that the above procedure starting with β' instead of β leads to an additional error at most $O(N^{-1})$. We have proved (65).

Eventually we show (66). In the proof of Theorem 4.2 we have seen that for even N we have the inclusion

$$H_N^{\frac{2}{N}} \subseteq T_N^{\frac{N}{2}+1}.$$

The upper bound $(N/2 + 1)/(N/2 + 2) = (N + 2)/(N + 4)$ follows from (5) again. \square

Theorem 4.8 is obtained in a similar way.

Proof of Theorem 4.8. Assume $\lambda > 0$ and ζ satisfies $\lambda_N(\zeta) > \lambda$. If we let $k = \lceil \lambda^{-1} \rceil$ then (25) implies $w_k(\zeta) > N - 2k + 1$. In other words $G_N^\lambda \subseteq T_k^{N-2k+1}$. Hence, when $N - 2k + 1 \geq k$ or equivalently $N \geq 3k - 1$, the formula (5) implies (67). The left lower bound in (68) rephrases (61). To prove the right bound, we again reverse the proof of Theorem 3.1. In the estimate (97) identify $N = m + n$. We see that if for $n \in \{1, 2, \dots, N\}$ and $C \geq 1$ the identity

$$\frac{Cn - (N - n)}{N - n + n(1 + C(n - 1))} = \lambda$$

holds, then $T_n^{Cn} \subseteq H_N^\lambda$. By (5), we further conclude

$$(114) \quad h_N^\lambda \geq t_n^{Cn} = \frac{n + 1}{nC + 1}.$$

In other words, we have

$$(115) \quad h_N^\lambda \geq \frac{n + 1}{nC + 1}, \quad \text{if } C = \frac{N(\lambda + 1) - n}{(\lambda + 1)n - \lambda n^2}.$$

Inserting the expression for C , after some simplification we derive

$$(116) \quad h_N^\lambda \geq \max_{1 \leq n \leq N} \frac{(1 + n)(1 + \lambda - n\lambda)}{(1 + \lambda)(N + 1 - n)}.$$

We may choose

$$n = \left\lfloor \frac{\lambda^{-1} + 1}{2} \right\rfloor = \left\lfloor \frac{\lambda + 1}{2\lambda} \right\rfloor,$$

since $N \geq \lambda^{-1} \geq (1 + \lambda)/(2\lambda) \geq n$ follows from the assumption $\lambda \leq 1$. Insertion in (115) yields (68). \square

We infer (69) with a refined treatment of (116), and (70) similarly to (65). We only sketch the proof.

Proof of Theorem 4.9. It can easily be checked that the right hand side maximum in (116) is obtained for the integer n if $\lambda \in I_n$, and yields the bounds (69) in the theorem. The estimate (70) can be derived with method of the proof of (65) and using the precise dimension formula for \mathcal{A} and \mathcal{B} defined in (111), we skip the computations. \square

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